SWERR-TR-72-37

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SOLUTION OF FLUID DYNAMIC EQUATIONS FOR GUN TUBE FLOW BY THE METHOD OF WEIGHTED RESIDUALS

PART I. UNSTEADY COMPRESSIBLE BOUNDRY LAYERS

TECHNICAL REPORT

Dr. Rao V. S. Yalamanchili
and
Philip D. Benzkofer

June 1972

RESEARCH DIRECTORATE

WEAPONS LABORATORY AT ROCK ISLAND

RESEARCH, DEVELOPMENT AND ENGINEERING DIRECTORATE

U. S. ARMY WEAPONS COMMAND

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130

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Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) U.S. Army Weapons Command Research, Dev. and Eng. Directorate Rock Island, Illinois 61201	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE
SOLUTION OF FLUID DYNAMIC EQUATIONS FOR GUN TUBE FLOW BY THE METHOD OF WEIGHTED RESIDUALS (U)
PART I. UNSTEADY COMPRESSIBLE BOUNDARY LAYERS

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

5. AUTHOR(S) (First name, middle initial, last name)
Dr. Rao V. S. Yalamanchili and Philip D. Benzkofer

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
June 1972	99	54

8a. CONTRACT OR GRANT NO. b. PROJECT NO. DA 1T061101A91A c. AMS Code 501A.11.844 d.	9a. ORIGINATOR'S REPORT NUMBER(S) SWERR-TR-72-37 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

10. DISTRIBUTION STATEMENT
Approved for public release, distribution unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	U. S. Army Weapons Command

13. ABSTRACT

The overall gun tube heat transfer problem applicable to any weapon, ammunition, and firing schedule was analyzed. Toward this goal, the solution to one of the five problems involved was investigated by personnel of the Research Directorate, Weapons Laboratory at Rock Island. The governing equations of unsteady compressible boundary layers are a system of nonlinear parabolic partial differential equations with three independent variables. The transverse coordinate is modified to absorb the compressibility effect. The stream function is introduced to satisfy the continuity and also to eliminate one of the dependent variables. The method of weighted residuals is used to reduce by one the number of independent variables. The functional in approximate solution form is chosen on the basis of the asymptotic solution of the steady differential equations for large values of the spacelike coordinate. The error functions, consequently, occur in the solution form for boundary layers. The method of Galerkin is used as the error-distribution principle. All integrations across the boundary layer are performed analytically. The Method of Lines is used to reduce the resulting partial differential equations in two independent variables to an approximate set of ordinary differential equations. This procedure enables one to solve for derivatives by the reduction of a matrix with elements not more than the number of undetermined parameters introduced into the solution form. Finally, the solutions are obtained by a fourth order Runga-Kutta integration scheme. The typical output contains not only profiles of velocity components, temperature, and density but also various boundary layer parameters and convective heat transfer coefficient. The numerical results are in agreement with Hall and Blasius results with even one term in approximate solution form. (U) (Yalamanchili, R.V.S. and Benzkofer, P.D.)

DD FORM 1 NOV 65 1473 REPLACES DD FORM 1473, 1 JAN 64, WHICH IS OBSOLETE FOR ARMY USE.

Unclassified
Security Classification

Unclassified
Security Classification

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
1. Gun Tube Heat Transfer						
2. Unsteady Compressible Boundary Layers						
3. Nonlinear Partial Differential Equations						
4. Method of Weighted Residuals						
5. Galerkin Method						
6. Method of Lines						

Unclassified
Security Classification

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CONTENTS

CONTENTS

NOMENCLATURE

A_1, A_2, B_1, B_2 = Functions of x and t as defined in Equation 3.11

C = Coefficient in dynamic viscosity-temperature relationship

C_f = Skin friction coefficient

C_p = Specific heat at constant pressure

D = Inner diameter of the tube

f's, g's = Functions in the ordinary differential equations

g = Gravitational constant

h = Convective heat transfer coefficient

K = Thermal conductivity

L = Length of the plate

N_u = Nusselt number

p = Pressure

Pr = Prandtl number

Pr_t = Turbulent Prandtl number

Q = Heat flux

r = Function of similarity variable (η)- Equation 5.12

R = Gas constant

Re = Reynolds number

s = Function of similarity variable (η) - Equation 5.12

NOMENCLATURE

St = Stanton number

t = Time

T - Temperature

u = Tangential velocity component

u^+ = Dimensionless tangential velocity - Equation 2.7

v = Normal velocity component

V = Velocity

x = Longitudinal coordinate

y = Transverse coordinate

$$\bar{y} = \int_0^y \frac{\rho}{\rho_o}\, dy$$

y^+ = Dimensionless transverse coordinate

$$\mathrm{erf}(b_1\bar{y}) = \frac{2}{\sqrt{\pi}} \int_0^{B_1\bar{y}} e^{-m^2} dm$$

ψ = Stream function

$\bar{\Psi}$ = Approximation to ψ in terms of ψ_1, ψ_2, ψ_3

θ = Difference in temperatures between gas and wall

ρ = Density of gas

NOMENCLATURE

γ = Specific heat ratio

τ = Shear stress

ν = Kinematic viscosity

ε = Eddy viscosity

μ = Dynamic viscosity

η = Covolume of propellant gas or similarity

ω = Exponent in dynamic viscosity-temperature relationship

$\delta_1, \delta_2, \Omega_1, \Omega_2$ = Coefficients defined in Equation 3.11

δ = Boundary layer thickness

δ_d = Displacement thickness

δ_{ed} = Energy-dissipation thickness

δ_n = Enthalpy thickness

δ_m = Momentum thickness

δ_u = Velocity thickness

χ = Dimensionless x-coordinate

Subscripts:

0 = Reference quantity

1 = Outer edge of the boundary layer

b = Behind the shock

w = Wall conditions

NOMENCLATURE

Superscripts:

- $\dot{}$ = Differentiation with respect to time
- $'$ = Differentiation with respect to similarity variable

1. INTRODUCTION

As the projectile in a weapon moves ahead because of the high-pressure gases created by the burning propellant, the propellant gas will be set into motion starting from rest. Since the governing equations of fluid dynamics for many problems of interest are a system of nonlinear partial differential equations which are dominated by real gas and non-equilibrium effects, no general solutions exist by which arbitrary initial and boundary conditions are allowed. Therefore, examination of the flow field and the subdivision of the overall problem by consideration of the dominant features only seem appropriate. The ultimate objective is to establish a capability to perform overall heat-transfer analysis for any given dimensions of the weapon and for specified propellant characteristics. Toward this goal, the propellant gas convective heat-transfer problem[1] is divided into five subproblems. (1) generation of thermochemical properties for any given propellant, (2) transient inviscid compressible flow through the gun barrel (core flow), (3) unsteady viscous compressible flow on the bore surface (boundary layers), (4) transient heat diffusion through the multilayer gun tube, (5) unsteady free convection and radiation outside the gun tube.

As the propellant burns, more and more hot gases will be generated. Typical gas in a chamber contains a temperature of 5000°R and a pressure of 50,000 psi. The thermochemistry of propellants involves determination of chemical composition of gases either by finite-rate chemistry or by chemical-equilibrium chemistry and the derivation of gas properties from the composition. Thermochemical properties of typical propellant gases are being predicted by use of a NASA-LEWIS thermochemical program.[2] The gases were highly toxic. Typical composition of the gases: M18 (CO = 0.41, H_2 = 0.19, H_2O = 0.16, N_2 = 0.1, CO_2 = 0.08) and IMR (CO = 0.43, H_2 = 0.12, H_2O = 0.21, N_2 = 0.12, CO_2 = 0.12). IMR is better than M18 as far as combustion is concerned, but IMR is still a long way from possible complete combustion. The consequences of incomplete combustion are muzzle flash, smoke, and fire in addition to low efficiency. With this program, one can compute not only chemical composition of gases but also specific heats, gas constant, and so forth as functions of pressure and temperature.

In instances of the unsteady core-flow problem, uniform density, uniform temperature, and linear velocity gradient are commonly used in a gas flow between the breech and the bullet. Since the governing equations of this problem are of the hyperbolic type, they can be solved by the well-known

method of characteristics. Some of the results obtained by this method were discussed[3] before. None of these assumptions is reasonable until after peak heating occurs. In general, however, the linear velocity gradient assumption is better than the uniform density assumption.

The boundary layer problem can also be interpreted as forced convective heat transfer in guns. The magnitude of convective heat transfer in guns is large primarily because of the high values of gas densities that exist and because of the large gas-to-wall temperature differences in addition to the larger gas flow velocities which constitute the convective heat transfer driving potential. Experimental data are commonly correlated with the various analytical or empirical techniques. The most popular method for heat transfer correlation purposes derives from the postulates of Nordheim, Soodak and Nordheim.[4] These authors theorize that the propellant gas wall shear friction factor is dependent upon only the gun surface roughness considerations. The justification for elimination of the Reynolds' number as a parameter is based on order of magnitude estimates of boundary layer thickness by use of laminar boundary layer considerations. The recommended form of the friction factor is thus dependent upon only the ratio of surface roughness to barrel radius. Consequently, the friction factor is assumed to be independent of space or time within a given gun barrel. Reynolds' analogy is subsequently used, and the heat transfer coefficient becomes simply proportional to gas density, velocity, and specific heat. Total calorimeter data have subsequently been rationalized in terms of the value of the friction factor that, with internal ballistic considerations, obtains the spatial variation of heat load to the gun barrel derived from a single shot. Example values of experimental friction factor ($C_f/2 = \tau_w/\rho V^2$) that were reported vary from .0011 to .0035. Geidt[5] assumes a value of .002 and finds that his measured heat flux is predicted generally within about a factor of 2. Other writers have interpreted pressure gradients within the gun barrel in an attempt to evaluate wall shear for the application of Reynolds' analogy for heat transfer.

Another approach for correlation of experimental data is based on the Dittus-Boetter[6] relation for steady, fully developed, turbulent pipe flow. This relation is also judged to be inaccurate in several references. Cornell Aeronautical Laboratory[7] proposed a combined analytical-experimental approach based on steady, fully developed, pipe flow concepts.

The state of the art in unsteady boundary layers and turbulent models is quite limited. A symposium[8] was held on

unsteady boundary layers at Laval University (1971) under the auspices of International Union of Theoretical and Applied Mechanics. However, their proceedings are still in press. Patel and Nash[9] discussed solutions of unsteady two-dimensional incompressible turbulent boundary layer equations by explicit finite-difference techniques Akamatsu[10] pointed out some kinds of unsteady boundary layers induced by shock waves in a shock tube. Woods[11] identified some industrial problems (such as pulse turbine, reciprocating engine, emergency blowdown of a chemical plant autoclave system, high-speed train entering a tunnel, and exhaust system of an internal combustion engine) associated with unsteady boundary layers. Foster[12] solved unsteady isothermal turbulent boundary layers with several approximations by the integral method and the method of characteristics. Anderson and Dahm[13] solved unsteady laminar boundary layer equations by the integral matrix procedure. Shelton[14] developed a solution procedure by combination of integral methods and finite-difference methods for turbulent boundary layers that are in compliance with Croco-Lees[51] relationship for temperature. At the end, the Chilton-Colburn[52] analogy was used to compute convective heat transfer coefficients.

Dahm and Anderson[15] formulated an analytical boundary layer procedure based on the compressible time-dependent boundary layer momentum integral equation by the simpler Croco-Lees relationship and the method of characteristics. Convective heat transfer was evaluated based on the Chilton-Colburn analogy of the energy boundary layer to the momentum boundary layer. Since the momentum and the energy boundary layer equations are dissimilar in an accelerating flow, an approximate solution of the energy integral equation is expected to yield better heat transfer results than applications of the Chilton-Colburn analogy to an approximate solution of the momentum integral equation

The solution of transient heat diffusion through gun barrel walls was established quite satisfactorily by analytical,[16] finite-difference[17] and finite-element[18,19,20] techniques. These are quite good for the purpose of establishing propellant gas convective heat transfer coefficients.

The unsteady free convection and radiation around gun tubes was discussed.[21] The radiation and convection contributions were of the same order of magnitude. The estimations based on pure convection show that the flow around the gun tube is still in the laminar range. Since the surface temperatures change quite rapidly for automatic weapons, the governing time-dependent nonlinear partial differential equations with three independent variables were solved by an explicit finite-difference scheme The stability criteria were

established by Von Neumann and Dusinberre[22] methods. The convective heat transfer coefficients can vary as much as 100 per cent from the minimum value because boundary layers are thinner on the lower half of the gun tube and thicker on the upper half of the cylindrical gun tube.

2. STATEMENT OF PROBLEM

The present investigation concerns primarily the formulation and the solution of transient viscous compressible flow on the bore surface. However, this is affected extensively by unsteady core flow and unsteady heat diffusion through the gun tube due to boundary conditions. The flow characteristics are unknown for gun barrel flows. The experimental data are lacking because of the moving bullet. This obstacle may be overcome provided one takes advantage of the similarities between the moving bullet (small mass) and a moving shock in a shock tube. Therefore, shock tube data should be collected and analyzed for possible use in predicting gun barrel flow characteristics.

As the propellant gases expand behind the projectile, a boundary layer forms at the breech end and thickens as the flow proceeds downstream. An unusual feature of the velocity boundary layer is that it disappears as the bullet is approached since all fluid at the base of the bullet must be moving at bullet velocity. Mathematically, this amounts to the requirement of an additional boundary condition at a downstream location. The numerical techniques applied to most boundary layer problems require the specification of profiles at the upstream end of the flow and allow a "marching" along the flow direction. For the usual time-dependent boundary layer problem, an initial condition to describe the boundary layer flow at time zero and boundary conditions as functions of time are necessary. No downstream condition is added. The question then logically arises as to how the boundary conditions, at both ends of the flow, can be satisfied in any one analysis. If the analysis is carried out from both ends of the flow, the results may not match anywhere in the middle of the flow. The compatible conditions are required before one can accept the results in the middle of the flow.

The laminar boundary layer becomes unstable if the Reynolds' number becomes sufficiently high, thus small disturbances will be amplified causing transition to a turbulent type of boundary layer. For a flat plate with zero pressure gradient, the laminar boundary layer has been experimentally shown to be quite stable for the length Reynolds' numbers Re_x upward to about 80,000, and this laminar boundary layer can extend to a Reynolds' number of

several million if the free-stream turbulence is very low and if the surface is very smooth. For engineering calculations unless other information is available, transition will be assumed generally to occur in the 200,000 to 500,000 range. These figures are only approximate and may be good for a smooth surface with a fair amount of free stream turbulence.

The length Reynolds' number Re_x is not very convenient for a transition criterion since it is based on a constant free stream velocity and may not be meaningful if it is allowed to vary with axial coordinate (x) such as the accelerating flow in a gun barrel In such circumstances, it is preferable to have a local parameter such as momentum thickness (δ_m) instead of x. If a critical Reynolds' number Re_x of about 360,000 is chosen as transition criterion for a constant free stream velocity, the equivalent criterion for accelerating flow becomes $Re_\delta = 0.664 \sqrt{Re_x} = 398.4$

The Reynolds' number, based on local distance, does not have any boundary layer characteristics, whereas the Reynolds' number based on momentum thickness does represent the important parameter of the boundary layer. Since transition to a turbulent boundary layer is dependent strongly upon the growth of the boundary layer, the Reynolds' number based on momentum thickness is logically considered to establish transition criteria.

The rate of heat transfer from the hot propellant gases to the barrel is controlled by development of the boundary layers. The flow in the gun barrel boundary layers could be laminar, transitional or turbulent in nature. The type of boundary layer at a particular cross section at any instant need not be the same as at another instant. Since the flow must start from rest and must also satisfy zero boundary layer thickness at the bullet base because of the scraping action of the bullet, laminar flow always exists in some parts of the gun barrel boundary layers

Flow in a laminar boundary layer will eventually become unstable as the Reynolds' number is increased. The boundary layer thickness, skin friction, and heat transfer increase more rapidly in turbulent flow than in laminar flow. The eddy viscosity is the dominating mechanism for such increases. The boundary layer flow can be turbulent somewhere in the middle of the flow between the breech and the bullet base. A transitional region should exist between the laminar and the turbulent regions. However, because of limited knowledge about transitional regions, the flow will be assumed to change suddenly from laminar to turbulent flow at a time and place

determined by the well-known laminar-turbulent transition criteria discussed above. Therefore, an analysis of the unsteady boundary layer is needed for laminar and turbulent boundary layers.

The analysis of unsteady compressible boundary layers on the bore surface is one of the most difficult problems due to the limited state of the art and also to the existence of laminar, transitional and turbulent regimes within the boundary layer. The governing equations of unsteady compressible boundary layers are a system of nonlinear partial differential equations of parabolic type with three independent variables. These are given below:

Continuity:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho v) = 0 \tag{2.1}$$

Momentum:

$$\rho \frac{\partial u}{\partial t} \div \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = - \frac{\partial p}{\partial x} + \frac{\partial}{\partial y} [\rho(\nu+\varepsilon)\frac{\partial u}{\partial y}] \tag{2.2}$$

Energy:

$$\rho C_p (\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y}) = \frac{\partial p}{\partial x} + u \frac{\partial p}{\partial x} + \rho(\nu+\varepsilon)(\frac{\partial u}{\partial y})^2$$

$$+ \frac{\partial}{\partial y} [(K + \frac{\rho C_p \varepsilon}{Pr_t}) \frac{\partial T}{\partial y}] \tag{2.3}$$

Equation of State:

$$p (\frac{1}{\rho} - \eta) = RT \tag{2.4}$$

Boundary Conditions:

$$\begin{aligned} y = o &: u = o,\ v = o,\ T = T_w(x, t) \\ y \approx \infty &: u = u_1(x, t),\ T = T_1(x, t) \\ x = x_o &: u = u_u(y, t),\ T = T_u(y,t) \end{aligned} \tag{2.5}$$

Dynamic Viscosity:

$$\mu = CT^{\omega} \tag{2.6}$$

The following eddy viscosity (ε) model may be used for turbulent boundary layers:

$$y^{+} = \frac{y\sqrt{g\ \tau_{\omega}/\rho}}{\nu}$$

$$u^{+} = \frac{u}{\sqrt{g\ \tau_{\omega}/\rho}} \tag{2.7}$$

$$\varepsilon/\nu = \frac{1-y/r_o}{du^{+}/dy^{+}}$$

$$\varepsilon/\nu = -\frac{y}{r_o} \quad \text{for } o < y^{+} < 5$$

$$\varepsilon/\nu = \frac{y^{+}(1-\frac{y}{r_o})}{5} - 1 \quad \text{for } 5 < y^{+} < 30$$

and $$\varepsilon/\nu = \frac{y+(1-\frac{y}{r_o})-1}{2.5} \quad \text{for } 30 < y^{+} < \infty$$

The objective is to obtain the dependent variables u, v and T as a function of x, y and t subjected to the boundary conditions given by Equation 2.5. This is discussed in sections 3 and 4.

3. THEORETICAL ANALYSIS

Various methods exist for the solution of steady boundary layer equations. The classical Von Karman-Pohlhausen integral method is popular because of the quickness and simplicity characteristics. For particular free-stream distributions such as flow similarity, the partial differential equations can be reduced to ordinary differential equations by similarity variables. At the end, the results are obtained by numerical integration. Instead of solving partly analytically and partly numerically, one can also solve partial differential equations of boundary layer by finite-difference schemes. As an alternative and also as a reduction in computational times, one can use the method of weighted residuals for the solution

of boundary layer equations. The last approach is used in the present investigation.

The dependent variables are u, v, and T. The independent variables are x, y, and t. One can easily reduce one of the dependent variables such as v by introduction of a stream function, ψ. The equation of continuity is satisfied automatically if the velocity components are denoted by the following equations:

$$u = \frac{\rho_o}{\rho}\frac{\partial\psi}{\partial y}$$

$$v = -\frac{\rho_o}{\rho}\left(\frac{\partial\psi}{\partial x} + \frac{\partial\bar{y}}{\partial t}\right) \tag{3.1}$$

$$\bar{y} = \int_0^y \frac{\rho}{\rho_o}\, dy$$

where ρ_o represents a reference density for nondimensional purposes.

The Howarth and Dorodnitsyn transformations among others are an important class of transformations specifically designed for compressible fluids. The purpose of these transformations is to remove the density from some of the boundary layer equations and to present the equations in a form closely resembling the incompressible boundary layer equations. After these investigations have been pursued, the transverse coordinate is modified to absorb the compressibility effect. The following equations can be obtained by the specializing of Equations 2.2 and 2.3 to the boundary layer edge conditions ($y \approx \infty$):

$$\rho_1 \frac{\partial u_1}{\partial t} + \rho_1 u_1 \frac{\partial u_1}{\partial x} = -\frac{\partial p}{\partial x}$$

$$\rho_1 C_p \left(\frac{\partial T_1}{\partial t} + u_1 \frac{\partial T_1}{\partial x}\right) = \frac{\partial p}{\partial t} + u_1 \frac{\partial p}{\partial x} \tag{3.2}$$

The momentum equation is reduced to the following form if Equations 3.1 and 3.2 are utilized in addition to the identity

$$\frac{\partial\psi}{\partial y} = \frac{\rho}{\rho_o}\frac{\partial\psi}{\partial\bar{y}} :$$

$$\frac{\partial}{\partial t}\left(\frac{\partial\psi}{\partial\bar{y}}\right) + \frac{\partial\psi}{\partial\bar{y}}\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial\bar{y}}\right) - \frac{\partial\psi}{\partial x}\frac{\partial}{\partial\bar{y}}\left(\frac{\partial\psi}{\partial\bar{y}}\right) - \frac{\partial\bar{y}}{\partial t}\frac{\partial}{\partial\bar{y}}\left(\frac{\partial\psi}{\partial\bar{y}}\right)$$

$$= \frac{\rho_1}{\rho}\frac{\partial u_1}{\partial t} + \frac{\rho_1}{\rho}u_1\frac{\partial u_1}{\partial x} + \frac{\partial}{\partial\bar{y}}\left[(\nu+\varepsilon)\frac{\partial}{\partial\bar{y}}\left(\frac{\partial\psi}{\partial\bar{y}}\right)\right] \qquad (3.3)$$

The pressure gradient terms in the energy equation can be expressed in terms of free-stream quantities:

$$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} = \rho_1 C_p\left(\frac{\partial T_1}{\partial t} + u\frac{\partial T_1}{\partial x}\right) - (u - u_1)\rho_1$$

$$\left(\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x}\right) \qquad (3.4)$$

This equation is obtained by use of Equation 3.2.

A new dependent variable can now be conveniently introduced for temperature difference as

$$\theta = (T - T_W) \qquad (3.5)$$

Substitution of Equations 3.1, 3.4, and 3.5 into the energy Equation 2.3 yields the following equation for the variable θ:

$$C_p\left[\frac{\partial\theta}{\partial t} + \frac{\partial\psi}{\partial\bar{y}}\frac{\partial\theta}{\partial x} - \left(\frac{\partial\psi}{\partial x} + \frac{\partial\bar{y}}{\partial t}\right)\frac{\partial\theta}{\partial\bar{y}}\right] + C_p\left(\frac{\partial T_W}{\partial t} + \frac{\partial\psi}{\partial\bar{y}}\frac{\partial T_W}{\partial x}\right)$$

$$= \frac{\partial}{\partial}\left[\frac{\rho}{\rho_o{}^2}\left(K + \frac{\rho C_p\varepsilon}{Pr_t}\right)\frac{\partial\theta}{\partial\bar{y}}\right] + (\nu+\varepsilon)\left(\frac{\rho}{\rho_o}\right)^2\frac{\partial^2\psi}{\partial\bar{y}^2}$$

$$+ \frac{\rho_1}{\rho}C_p\left(\frac{\partial T_1}{\partial t} + u_1\frac{\partial T_1}{\partial x}\right) - (u - u_1)\frac{\rho_1}{\rho}\left(\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x}\right) \qquad (3.6)$$

Further analysis of Equations 3.3 and 3.6 will be conducted in the remainder of this section.

3.1 Method of Weighted Residuals

The method of weighted residuals unifies many approximate method' of the solution of differential equations that are in use day. For unsteady heat conduction, the finite-element method and the usual finite-difference method were shown[23,18,19] to be special instances of the method of weighted residuals with a general weighting function. In References 24 and 25 in a more formal way, variational principles proposed by several authors are all applications of the method of weighted residuals.

An excellent review (187 references) on the method of weighted residuals was presented by Finlayson and Scriven.[25] In literature, this technique is commonly called the error distribution-principle. Usually, the method of weighted residuals is used to reduce by one the number of independent variables in any system of partial differential equations. However, some exceptions exist. For example, Kaplan and Bewick[26] and Kaplan and Marlowe[27] used the method of weighted residuals to reduce the number of independent variables from four to two. When this procedure is combined with other numerical methods, significant reductions in computer time to obtain a solution is apparent.

The basic steps that are involved in any application of the method of weighted residuals are given below. Equation 3.3 can be abbreviated as

$$\psi\ (x,\overline{y},t) = 0 \tag{3.7}$$

Let the following equation be an approximate solution chosen to represent ψ. The selection of approximating functions ψ_1, ψ_2, ψ_3 will be discussed later. However, these are usually chosen to satisfy any known boundary or initial conditions with respect to the independent variable to be removed.

$$\psi \simeq \overline{\psi} = \psi_1\ (B_1(x,t),y) + \psi_2(B_2(x,t),\overline{y}) + \psi_3(B_3(x,t),\overline{y}) \tag{3.8}$$

Substitution of the approximate solution for ψ in Equation 3.7 yields the following relationship for the residual error:

$$\psi_1(B_1(x,t),\overline{y}) + \psi_2(B_2(x,t),\overline{y}) + \psi_3(B_3(x,t),\overline{y}) =$$

$$R(B_1,B_2,B_3,\overline{y}) \tag{3.9}$$

The functions B_1, B_2, and B_3 are to be determined such that, in some sense, the residual error approaches zero. This objective can be accomplished by multiplication of the approximate Equation 3.9 by a set of three linearly independent weighting functions (W_1, W_2, W_3) that are dependent upon B's and $\bar{y}$. The weighted residual error is then integrated over $\bar{y}$ between o and ∞, and the result is set equal to zero.

i.e.,

$$\int_0^\infty R(B_1,B_2,B_3,\bar{y})\ W_1(B_1(x,t)\bar{y})\ d\bar{y} = 0$$

$$\int_0^\infty R(B_1,B_2,B_3,\bar{y})\ W_2(B_2(x,t),\bar{y})\ d\bar{y} = 0$$

$$\int_0^\infty R(B_1,B_2,B_3,\bar{y})\ W_3(B_3(x,t),\bar{y})\ d\bar{y} = 0 \qquad (3.10)$$

The resulting equations contain the unknown functions B_1, B_2, and B_3 with independent variables only x and t. Thus, the objective of method of weighted residuals is achieved by removal of the dependency of one of the independent variables ($\bar{y}$). In return, a set of equations (Equation 3.10) is derived for the determination of B_1, B_2, and B_3. The approximate solution (Equation 3.8) need not be limited to three terms. In fact, increasing the number of terms in the approximate solution increases the accuracy. However, three terms may be enough for engineering accuracy.

3.1.1 Approximate Solution

The choice of approximating functions in an assumed solution form is crucial in applying the method of weighted

residuals. No way presently seems to be available to select the approximating functions systematically for all problems. Selection of approximating functions remains somewhat dependent upon the user's intuition and experience, and this is often regarded as a major disadvantage of method of weighted residuals. Crandall[28] stated that the variation between results obtained by application of different weighting functions to the same approximate solution is much less significant than the variations that can result from the choice of different approximate solutions. Sometimes, one can obtain the exact solution by use of method of weighted residuals if the right choice is made in the selection of the approximate solution form.

The selection of approximating functions is still a definite problem even though the method of weighted residuals has existed for more than 50 years. Several sets of approximating functions that satisfy boundary conditions are permissible, and to choose one as the best is impossible. The usual approach of selecting the approximating functions is based on satisfying the governing differential equation on the boundary in addition to satisfying the boundary conditions. However, Lowe[29] established the form of the approximating function in the following manner:

If integration in the unbounded domain is to be accomplished, then the integral must exist for large values of the argument. Therefore, the asymptotic expansion or variation of the dependent variables should be established for large values of the space-like variables. This primarily means that the exponential order of the dependent variables for large values of the space-like variables should be determined. The approximating functions should be chosen to exhibit that some exponential order and, if possible, the complete asymptotic order. At the end, the approximating functions should be made to satisfy the boundary conditions and, in addition, these approximating functions should satisfy the governing equation at the boundary. In effect, the residual will start and end at zero. The residual in the interior of the domain will then be adjusted by some error distribution-principle.

This is the approach used in the present investigation. Following this approach, one can see that the error function appears to be a fundamental form for approximating functions for forced flow boundary-layer problems. Therefore, the following approximate solution forms were respectively chosen for the dependent variables u (or ψ) and θ.

$$\frac{u}{u_1} = \frac{\partial\psi}{\partial\bar{y}} = \Omega_1 \text{ erf } (B_1\bar{y}) + \Omega_2 \text{ erf } (B_2\bar{y}) + \cdots$$

$$\frac{\theta}{\theta_1} = \delta_1 \text{ erf } (A_1\bar{y}) + \delta_2 \text{ erf } (A_2\bar{y}) + \cdots \qquad (3.11)$$

Where A's and B's are unknown functions of the other two independent variables x and t. The following must be true to satisfy the boundary conditions for large space-like variable.

$$\Omega_1 + \Omega_2 + \cdots = 1$$

$$\delta_1 + \delta_2 + \cdots = 1 \qquad (3.12)$$

3.1.2 Weighting Functions

With the weighting functions, various criteria that are available are determined for the distribution of error in the method of weighted residuals. Basically, five criteria are available. The origin of these techniques and corresponding weighting functions are given in the following table:

Method	Origin	Weighting Function
Collocation	Frazer, Jones and Skan[30]	Divac - Delta
Galerkin	Galerkin[31]	$\frac{\partial\bar{\psi}}{\partial B}$ (Equation (3.8) $i = 1,2,3$
Method of Moments	Kravchuk[32]	any complete set of functions $(1, y, y^2 \cdots)$
Method of Least Squares	Picone[33]	$\frac{\partial R}{\partial B_i}$ (Equation (3.9)
Subdomain	Biezeno and Koch[34]	Unity in that subdomain and zero elsewhere

To name a particular criterion as a best one is impossible. The choice may depend upon the problem to be solved, the assumed approximate solution form, the number of terms in it, and also the parameter of interest in that problem. The following example concerns the Blasius problem of forced flow over a flat plate [$f''' + ff''/2 = 0$, $f(0) = 0$, the assumed solution form $f' = \text{erf}(B\eta)$].

	Dimensionless Wall Shear		Dimensionless Displacement Thickness	
Method	$f''(0)$	% error	δ_1	% error
Exact	.33206	0	1.7208	0
Galerkin	.3532	6.29	1.803	4.66
Method of Moments	.3631	9.33	1.753	1.87
Subdomain	.3631	9.33	1.753	1.87
Collocation	.3927	18.25	1.621	5.81

The results by use of the method of least squares is unavailable because of the possibility of imaginary results for similar classes of problems. Moreover, quite complex weighting functions will result by this technique. The parameter of interest for heat-transfer purposes is that of wall shear. The Galerkin method resulted in least error for wall shear. Finlayson and Scriven[24] solved a convective transport problem and concluded that the Galerkin method predicted exactly the same expression for Nusselt number other than a proportionality constant. In the instance of the Galerkin method, the number of terms in an assumed approximate solution did not yield significant differences in heat-transfer results. These characteristics provide a more desirable method for coupled, nonlinear, and complex partial differential equations. From the examples cited, the selection of criteria is evidently unimportant. However, one should carefully consider the selection of functions in the assumed approximate solution form.

On the basis of the examples cited above and the statements by Ames,[35] Finlayson and Scriven,[25] Schetz,[36] Snyder, Spriggs, and Stewart,[37] and Kaplan,[38] the Galerkin method is chosen for the present investigation.

No evidence is available to prove that the method is a superior technique for general problems. However, the Galerkin method was related, before, to variational calculus, and several proofs of convergence have been made for specific applications.

The weighting functions (Galerkin) for assumed solution forms (Equation 3.11) can be written as

$$\frac{2\bar{y}}{\sqrt{\pi}} \Omega_1 e^{-B_1^2 \bar{y}^2}$$

$$\frac{2\bar{y}}{\sqrt{\pi}} \Omega_2 e^{-B_2^2 \bar{y}^2}$$

$$\frac{2y}{\sqrt{\pi}} \delta_1 e^{-A_1^2 y^2}$$

$$\frac{2y}{\sqrt{\pi}} \delta_2 e^{-A_2^2 y^2} \qquad (3.13)$$

The first two weighting functions are meant for the momentum equation, whereas the last two pertain to the energy equation.

3.1.3 Integral Equations

The analysis mentioned under the method of weighted residuals can now be performed. The approximate solution forms (Equation 3.11) and the weighting functions (Equation 3.13) are already obtained. The objective of this section is to explain the transformation of the dependent variables from ψ and θ to B_1, B_2, A_1, and A_2. The objective also involves the removal of the transverse independent variable $\bar{y}$ from the transformed governing Equations 3.3 and 3.6. The stream function ψ and some of its derivatives must be evaluated before one can apply the method of weighted residuals to the momentum Equation 3.3.

$$\psi = \int \frac{\partial \psi}{\partial \bar{\eta}} \, d\bar{\eta}$$

$$= u_1 \Omega_1 \left(\bar{\eta} \, \mathrm{erf}(B_1 \bar{\eta}) + \frac{1}{B_1 \sqrt{\pi}} e^{-B_1^2 \bar{\eta}^2} + C_1 \right) + u_1 \Omega_2 \left(\bar{\eta} \, \mathrm{erf}(B_2 \bar{\eta}) + \frac{1}{B_2 \sqrt{\pi}} e^{-B_2^2 \bar{\eta}^2} + C_2 \right) \qquad [3.14]$$

where C_1 and C_2 are determined by applying known conditions on $\bar{y}$ to $\psi(\bar{y}=0, \psi=0)$ Knowing ψ, one can obtain

$$\frac{\partial \psi}{\partial x} = \Omega_1 \bar{\eta} \frac{\partial u_1}{\partial x} \mathrm{erf}(B_1 \bar{\eta}) + \Omega_2 \bar{\eta} \frac{\partial u_1}{\partial x} \mathrm{erf}(B_2 \bar{\eta}) + \frac{e^{-B_1^2 \bar{\eta}^2}}{\sqrt{\pi}} \left(2 u_1 \Omega_1 \bar{\eta}^2 \frac{\partial B_1}{\partial x} + \frac{\Omega_1}{B_1} \frac{\partial u_1}{\partial x} - \frac{\Omega_1}{B_1^2} u_1 \frac{\partial B_1}{\partial x} - 2 \Omega_1 \bar{\eta}^2 u_1 \frac{\partial B_1}{\partial x} \right)$$

$$+ \frac{e^{-B_2^2 \bar{\eta}^2}}{\sqrt{\pi}} \left(2 \Omega_2 \bar{\eta}^2 u_1 \frac{\partial B_2}{\partial x} + \frac{\Omega_2}{B_2} \frac{\partial u_1}{\partial x} - \frac{\Omega_2}{B_2^2} u_1 \frac{\partial B_2}{\partial x} - 2 \Omega_2 \bar{\eta}^2 u_1 \frac{\partial B_2}{\partial x} \right) - \frac{1}{\sqrt{\pi}} \left(\frac{\Omega_1}{B_1} \frac{\partial u_1}{\partial x} - \frac{\Omega_1}{B_1^2} u_1 \frac{\partial B_1}{\partial x} + \frac{\Omega_2}{B_2} \frac{\partial u_1}{\partial x} - \frac{\Omega_2}{B_2^2} u_1 \frac{\partial B_2}{\partial x} \right)$$

$$[3.15]$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \psi}{\partial \bar{\eta}} \right) = \Omega_1 \frac{\partial u_1}{\partial t} \mathrm{erf}(B_1 \bar{\eta}) + \frac{2}{\sqrt{\pi}} \Omega_1 \bar{\eta} u_1 \frac{\partial B_1}{\partial t} e^{-B_1^2 \bar{\eta}^2} + \frac{2}{\sqrt{\pi}} \Omega_1 \frac{\partial \bar{\eta}}{\partial t} u_1 B_1 e^{-B_1^2 \bar{\eta}^2} + \Omega_2 \frac{\partial u_1}{\partial t} \mathrm{erf}(B_2 \bar{\eta}) + \frac{2}{\sqrt{\pi}} \Omega_2 \bar{\eta} u_1 \frac{\partial B_2}{\partial t} e^{-B_2^2 \bar{\eta}^2}$$

$$+ \frac{2}{\sqrt{\pi}} \Omega_2 \frac{\partial \bar{\eta}}{\partial t} u_1 B_2 e^{-B_2^2 \bar{\eta}^2} \qquad [3.16]$$

$$\left(\frac{\partial \psi}{\partial \bar{\eta}} \right) \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial \bar{\eta}} \right) = \Omega_1^2 u_1 \frac{\partial u_1}{\partial x} \mathrm{erf}^2(B_1 \bar{\eta}) + \frac{2}{\sqrt{\pi}} \Omega_1^2 u_1^2 \bar{\eta} \frac{\partial B_1}{\partial x} \mathrm{erf}(B_1 \bar{\eta}) e^{-B_1^2 \bar{\eta}^2} + \Omega_1 \Omega_2 u_1 \frac{\partial u_1}{\partial x} \mathrm{erf}(B_1 \bar{\eta}) \mathrm{erf}(B_2 \bar{\eta})$$

$$+ \frac{2}{\sqrt{\pi}} \Omega_1 \Omega_2 u_1^2 \bar{\eta} \frac{\partial B_2}{\partial x} \mathrm{erf}(B_1 \bar{\eta}) e^{-B_2^2 \bar{\eta}^2} + \Omega_1 \Omega_2 u_1 \frac{\partial u_1}{\partial x} \mathrm{erf}(B_1 \bar{\eta}) \mathrm{erf}(B_2 \bar{\eta}) + \frac{2}{\sqrt{\pi}} \Omega_1 \Omega_2 u_1^2 \bar{\eta} \frac{\partial B_1}{\partial x} \mathrm{erf}(B_2 \bar{\eta}) e^{-B_1^2 \bar{\eta}^2}$$

$$+ \Omega_2^2 u_1 \frac{\partial u_1}{\partial x} \mathrm{erf}^2(B_2 \bar{\eta}) + \frac{2}{\sqrt{\pi}} \Omega_2^2 u_1^2 \bar{\eta} \frac{\partial B_2}{\partial x} \mathrm{erf}(B_2 \bar{\eta}) e^{-B_2^2 \bar{\eta}^2} \qquad [3.17]$$

$$\left(\frac{\partial \psi}{\partial x} \right) \frac{\partial}{\partial \bar{\eta}}\left(\frac{\partial \psi}{\partial \bar{\eta}} \right) = \Big\{ \Omega_1 \bar{\eta} \frac{\partial u_1}{\partial x} \mathrm{erf}(B_1 \bar{\eta}) + \Omega_2 \bar{\eta} \frac{\partial u_1}{\partial x} \mathrm{erf}(B_2 \bar{\eta}) + \frac{1}{\sqrt{\pi}} e^{-B_1^2 \bar{\eta}^2} \left[2 \Omega_1 u_1 \bar{\eta}^2 \frac{\partial B_1}{\partial x} + \frac{\Omega_1}{B_1} \frac{\partial u_1}{\partial x} - \Omega_1 \frac{u_1}{B_1^2} \frac{\partial B_1}{\partial x} - 2 \Omega_1 u_1 \bar{\eta}^2 \frac{\partial B_1}{\partial x} \right]$$

$$+ \frac{1}{\sqrt{\pi}} e^{-B_2^2 \bar{\eta}^2} \left[2 \Omega_2 u_1 \bar{\eta}^2 \frac{\partial B_2}{\partial x} + \frac{\Omega_2}{B_2} \frac{\partial u_1}{\partial x} - \Omega_2 \frac{u_1}{B_2^2} \frac{\partial B_2}{\partial x} - 2 \Omega_2 u_1 \bar{\eta}^2 \frac{\partial B_2}{\partial x} \right] - \frac{1}{\sqrt{\pi}} \Big[\frac{\Omega_1}{B_1} \frac{\partial u_1}{\partial x} - \frac{\Omega_1 u_1}{B_1^2} \frac{\partial B_1}{\partial x}$$

$$+ \frac{\Omega_2}{B_2} \frac{\partial u_1}{\partial x} - \Omega_2 \frac{u_1}{B_2^2} \frac{\partial B_2}{\partial x} \Big] \Big\} \cdot \frac{2}{\sqrt{\pi}} u_1 \left[\Omega_1 B_1 e^{-B_1^2 \bar{\eta}^2} + \Omega_2 B_2 e^{-B_2^2 \bar{\eta}^2} \right] \qquad [3.18]$$

$$\frac{\partial \bar{\eta}}{\partial t} \frac{\partial}{\partial \bar{\eta}}\left(\frac{\partial \psi}{\partial \bar{\eta}} \right) = \frac{2}{\sqrt{\pi}} \Omega_1 u_1 B_1 \frac{\partial \bar{\eta}}{\partial t} e^{-B_1^2 \bar{\eta}^2} + \frac{2}{\sqrt{\pi}} \Omega_2 u_1 B_2 \frac{\partial \bar{\eta}}{\partial t} e^{-B_2^2 \bar{\eta}^2} \qquad [3.19]$$

$$\psi = \int \frac{\partial \psi}{\partial \bar{y}} \, d\bar{y}$$

$$= u_1 \Omega_1 \left(\bar{y}\, \mathrm{erf}(B_1 \bar{y}) + \frac{1}{B_1\sqrt{\pi}} e^{-B_1^2 \bar{y}^2} + C_1\right) + u_1 \Omega_2 \left(\bar{y}\, \mathrm{erf}(B_2 \bar{y}) + \frac{1}{B_2\sqrt{\pi}} e^{-B_2^2 \bar{y}^2} + C$$

where C_1 and C_2 are determined by applying known conditions on $\bar{y}$ to $\psi(\bar{y}=0, \psi=0$

$$\frac{\partial \psi}{\partial x} = \Omega_1 \bar{y} \frac{\partial u_1}{\partial x} \mathrm{erf}(B_1 \bar{y}) + \Omega_2 \bar{y} \frac{\partial u_1}{\partial x} \mathrm{erf}(B_2 \bar{y}) + \frac{e^{-B_1^2 \bar{y}^2}}{\sqrt{\pi}} \left(2 u_1 \Omega_1 \bar{y}^2 \frac{\partial B_1}{\partial x} + \frac{\Omega_1}{B_1} \frac{\partial u_1}{\partial x} - \frac{\Omega_1}{B_1^2}\right.$$

$$+ \frac{e^{-B_2^2 \bar{y}^2}}{\sqrt{\pi}} \left(2 \Omega_2 \bar{y}^2 u_1 \frac{\partial B_2}{\partial x} + \frac{\Omega_2}{B_2} \frac{\partial u_1}{\partial x} - \frac{\Omega_2}{B_2^2} u_1 \frac{\partial B_2}{\partial x} - 2 \Omega_2 \bar{y}^2 u_1 \frac{\partial B_2}{\partial x}\right) - \frac{1}{\sqrt{\pi}} \left(\frac{\Omega_1}{B_1}\right.$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \psi}{\partial \bar{y}}\right) = \Omega_1 \frac{\partial u_1}{\partial t} \mathrm{erf}(B_1 \bar{y}) + \frac{2}{\sqrt{\pi}} \Omega_1 \bar{y}\, u_1 \frac{\partial B_1}{\partial t} e^{-B_1^2 \bar{y}^2} + \frac{2}{\sqrt{\pi}} \Omega_1 \frac{\partial \bar{y}}{\partial t} u_1 B_1 e^{-B_1^2 \bar{y}^2} + \Omega_2 \frac{\partial}{\partial}$$

$$+ \frac{2}{\sqrt{\pi}} \Omega_2 \frac{\partial \bar{y}}{\partial t} u_1 B_2 e^{-B_2^2 \bar{y}^2}$$

$$\left(\frac{\partial \psi}{\partial \bar{y}}\right) \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial \bar{y}}\right) = \Omega_1^2 u_1 \frac{\partial u_1}{\partial x} \mathrm{erf}^2(B_1 \bar{y}) + \frac{2}{\sqrt{\pi}} \Omega_1^2 u_1^2 \bar{y} \frac{\partial B_1}{\partial x} \mathrm{erf}(B_1 \bar{y}) e^{-B_1^2 \bar{y}^2} + \Omega_1 \Omega_2$$

$$+ \frac{2}{\sqrt{\pi}} \Omega_1 \Omega_2 u_1^2 \bar{y} \frac{\partial B_2}{\partial x} \mathrm{erf}(B_1 \bar{y}) e^{-B_2^2 \bar{y}^2} + \Omega_1 \Omega_2 u_1 \frac{\partial u_1}{\partial x} \mathrm{erf}(B_1 \bar{y})$$

$$+ \Omega_2^2 u_1 \frac{\partial u_1}{\partial x} \mathrm{erf}^2(B_2 \bar{y}) + \frac{2}{\sqrt{\pi}} \Omega_2^2 u_1^2 \bar{y} \frac{\partial B_2}{\partial x} \mathrm{erf}(B_2 \bar{y}) e^{-B_2^2 \bar{y}^2}$$

$$\left(\frac{\partial \psi}{\partial x}\right) \frac{\partial}{\partial \bar{y}}\left(\frac{\partial \psi}{\partial \bar{y}}\right) = \left\{\Omega_1 \bar{y} \frac{\partial u_1}{\partial x} \mathrm{erf}(B_1 \bar{y}) + \Omega_2 \bar{y} \frac{\partial u_1}{\partial x} \mathrm{erf}(B_2 \bar{y}) + \frac{1}{\sqrt{\pi}} e^{-B_1^2 \bar{y}^2} \left[2 \Omega_1 u_1\right.\right.$$

$$+ \frac{1}{\sqrt{\pi}} e^{-B_2^2 \bar{y}^2} \left[2 \Omega_2 u_1 \bar{y}^2 \frac{\partial B_2}{\partial x} + \frac{\Omega_2}{B_2} \frac{\partial u_1}{\partial x} - \Omega_2 \frac{u_1}{B_2^2} \frac{\partial B_2}{\partial x} - 2 \Omega_2 u_1 \bar{y}\right.$$

$$\left.\left. + \frac{\Omega_2}{B_2} \frac{\partial u_1}{\partial x} - \Omega_2 \frac{u_1}{B_2^2} \frac{\partial B_2}{\partial x}\right]\right\} \cdot \frac{2}{\sqrt{\pi}} u_1 \left[\Omega_1 B_1 e^{-B_1^2 \bar{y}^2} + \Omega_2 B_2 e^{-B_2^2 \bar{y}^2}\right.$$

$$\frac{\partial \bar{y}}{\partial t} \frac{\partial}{\partial \bar{y}}\left(\frac{\partial \psi}{\partial \bar{y}}\right) = \frac{2}{\sqrt{\pi}} \Omega_1 u_1 B_1 \frac{\partial \bar{y}}{\partial t} e^{-B_1^2 \bar{y}^2} + \frac{2}{\sqrt{\pi}} \Omega_2 u_1 B_2 \frac{\partial \bar{y}}{\partial t} e^{-B_2^2 \bar{y}^2}$$

$$) + u_1\Omega_2\left(\bar{y}\,\mathrm{erf}(B_2\bar{y}) + \frac{1}{B_2\sqrt{\pi}}\,e^{-B_2^2\bar{y}^2} + C_2\right) \qquad [3.14]$$

plying known conditions on $\bar{y}$ to $\psi(\bar{y}=0, \psi=0)$ Knowing ψ, one can obtain

$$B_2\bar{y}) + \frac{e^{-B_1^2\bar{y}^2}}{\sqrt{\pi}}\left(2u_1\Omega_1\bar{y}^2\frac{\partial B_1}{\partial x} + \frac{\Omega_1}{B_1}\frac{\partial u_1}{\partial x} - \frac{\Omega_1}{B_1^2}u_1\frac{\partial B_1}{\partial x} - 2\Omega_1\bar{y}^2 u_1\frac{\partial B_1}{\partial x}\right)$$

$$- \frac{\Omega_2}{B_2^2}u_1\frac{\partial B_2}{\partial x} - 2\Omega_2\bar{y}^2 u_1\frac{\partial B_2}{\partial x}\right) - \frac{1}{\sqrt{\pi}}\left(\frac{\Omega_1}{B_1}\frac{\partial u_1}{\partial x} - \frac{\Omega_1}{B_1^2}u_1\frac{\partial B_1}{\partial x} + \frac{\Omega_2}{B_2}\frac{\partial u_1}{\partial x} - \frac{\Omega_2}{B_2^2}u_1\frac{\partial B_2}{\partial x}\right) \qquad [3.15]$$

$$\frac{\partial B_1}{\partial t}e^{-B_1^2\bar{y}^2} + \frac{2}{\sqrt{\pi}}\Omega_1\frac{\partial\bar{y}}{\partial t}u_1 B_1 e^{-B_1^2\bar{y}^2} + \Omega_2\frac{\partial u_1}{\partial t}\mathrm{erf}(B_2\bar{y}) + \frac{2}{\sqrt{\pi}}\Omega_2\bar{y}\,u_1\frac{\partial B_2}{\partial t}e^{-B_2^2\bar{y}^2} \qquad [3.16]$$

$$\frac{2}{\sqrt{\pi}}\Omega_1^2 u_1^2\bar{y}\frac{\partial B_1}{\partial x}\mathrm{erf}(B_1\bar{y})e^{-B_1^2\bar{y}^2} + \Omega_1\Omega_2 u_1\frac{\partial u_1}{\partial x}\mathrm{erf}(B_1\bar{y})\mathrm{erf}(B_2\bar{y})$$

$$f(B_1\bar{y})e^{-B_2^2\bar{y}^2} + \Omega_1\Omega_2 u_1\frac{\partial u_1}{\partial x}\mathrm{erf}(B_1\bar{y})\mathrm{erf}(B_2\bar{y}) + \frac{2}{\sqrt{\pi}}\Omega_1\Omega_2 u_1^2\bar{y}\frac{\partial B_1}{\partial x}\mathrm{erf}(B_2\bar{y})e^{-B_1^2\bar{y}^2}$$

$$\bar{y}) + \frac{2}{\sqrt{\pi}}\Omega_2^2 u_1^2\bar{y}\frac{\partial B_2}{\partial x}\mathrm{erf}(B_2\bar{y})e^{-B_2^2\bar{y}^2} \qquad [3.17]$$

$$\Omega_2\bar{y}\frac{\partial u_1}{\partial x}\mathrm{erf}(B_2\bar{y}) + \frac{1}{\sqrt{\pi}}e^{-B_1^2\bar{y}^2}\left[2\Omega_1 u_1\bar{y}^2\frac{\partial B_1}{\partial x} + \frac{\Omega_1}{B_1}\frac{\partial u_1}{\partial x} - \Omega_1\frac{u_1}{B_1^2}\frac{\partial B_1}{\partial x} - 2\Omega_1 u_1\bar{y}^2\frac{\partial B_1}{\partial x}\right]$$

$$\bar{y}^2\frac{\partial B_2}{\partial x} + \frac{\Omega_2}{B_2}\frac{\partial u_1}{\partial x} - \Omega_2\frac{u_1}{B_2^2}\frac{\partial B_2}{\partial x} - 2\Omega_2 u_1\bar{y}^2\frac{\partial B_2}{\partial x}\right] - \frac{1}{\sqrt{\pi}}\left[\frac{\Omega_1}{B_1}\frac{\partial u_1}{\partial x} - \frac{\Omega_1 u_1}{B_1^2}\frac{\partial B_1}{\partial x}\right.$$

$$\left.\right]\Big\}\cdot\frac{2}{\sqrt{\pi}}u_1\left[\Omega_1 B_1 e^{-B_1^2\bar{y}^2} + \Omega_2 B_2 e^{-B_2^2\bar{y}^2}\right] \qquad [3.18]$$

$$\frac{2}{\sqrt{\pi}}\Omega_2 u_1 B_2\frac{\partial\bar{y}}{\partial t}e^{-B_2^2\bar{y}^2} \qquad [3.19]$$

$$\frac{\rho_1}{\rho}\frac{\partial u_1}{\partial t} + \frac{\rho_1}{\rho} u_1 \frac{\partial u_1}{\partial x} = \frac{R\theta + \eta P + RT_w}{R\theta_1 + \eta P + RT_w}\frac{\partial u_1}{\partial t} + \frac{R\theta + \eta P + RT_w}{R\theta_1 + \eta P + RT_w} u_1 \frac{\partial u_1}{\partial x} \qquad [3.20]$$

$$\frac{\partial}{\partial \bar{\eta}}\left[(\nu + \epsilon)\frac{\partial}{\partial \bar{\eta}}\left(\frac{\partial \gamma}{\partial \bar{\eta}}\right)\right] = \frac{\partial}{\partial \bar{\eta}}\left[(\nu + \epsilon)\frac{2}{\sqrt{\pi}} u_1\left(\Omega_1 B_1 e^{-B_1^2\bar{\eta}^2} + \Omega_2 B_2 e^{-B_2^2\bar{\eta}^2}\right)\right] \qquad [3.21]$$

Combining all terms of the Momentum Equation and forming the residual, one obtains

$$\Omega_1 \frac{\partial u_1}{\partial t} \operatorname{erf}(B_1\bar{\eta}) + \frac{2}{\sqrt{\pi}}\Omega_1 u_1 \bar{\eta}\frac{\partial B_1}{\partial t} e^{-B_1^2\bar{\eta}^2} + \frac{2}{\sqrt{\pi}}\Omega_1 u_1 B_1 \frac{\partial \bar{\eta}}{\partial t} e^{-B_1^2\bar{\eta}^2} + \Omega_2 \frac{\partial u_1}{\partial t}\operatorname{erf}(B_2\bar{\eta}) + \frac{2}{\sqrt{\pi}}\Omega_2 u_1 \bar{\eta}\frac{\partial B_2}{\partial t} e^{-B_2^2\bar{\eta}^2}$$

$$+ \frac{2}{\sqrt{\pi}}\Omega_2 u_1 B_2 \frac{\partial \bar{\eta}}{\partial t} e^{-B_2^2\bar{\eta}^2} + \Omega_1^2 u_1 \frac{\partial u_1}{\partial x}\operatorname{erf}^2(B_1\bar{\eta}) + \frac{2}{\sqrt{\pi}}\Omega_1^2 u_1^2 \bar{\eta}\frac{\partial B_1}{\partial x}\operatorname{erf}(B_1\bar{\eta}) e^{-B_1^2\bar{\eta}^2} + 2\Omega_1\Omega_2 u_1 \frac{\partial u_1}{\partial x}\operatorname{erf}(B_1\bar{\eta})\operatorname{erf}(B_2\bar{\eta})$$

$$+ \frac{2}{\sqrt{\pi}}\Omega_1\Omega_2 u_1^2 \bar{\eta}\frac{\partial B_2}{\partial x}\operatorname{erf}(B_1\bar{\eta}) e^{-B_2^2\bar{\eta}^2} + \frac{2}{\sqrt{\pi}}\Omega_1\Omega_2 u_1^2 \bar{\eta}\frac{\partial B_1}{\partial x}\operatorname{erf}(B_2\bar{\eta}) e^{-B_1^2\bar{\eta}^2} + \Omega_2^2 u_1 \frac{\partial u_1}{\partial x}\operatorname{erf}^2(B_2\bar{\eta}) + \frac{2}{\sqrt{\pi}}\Omega_2^2 u_1 \bar{\eta}\frac{\partial B_2}{\partial x}\operatorname{erf}(B_2\bar{\eta}) e^{-B_2^2\bar{\eta}^2}$$

$$- \frac{2}{\sqrt{\pi}}\Omega_1^2 u_1 \bar{\eta}\frac{\partial u_1}{\partial x} B_1 \operatorname{erf}(B_1\bar{\eta}) e^{-B_1^2\bar{\eta}^2} - \frac{2}{\sqrt{\pi}}\Omega_1\Omega_2 u_1 \bar{\eta}\frac{\partial u_1}{\partial x} B_1 \operatorname{erf}(B_2\bar{\eta}) e^{-B_1^2\bar{\eta}^2} - \left(\frac{2}{\sqrt{\pi}}\right)^2 \Omega_1^2\Omega_2^2 u_1^2 \bar{\eta}^2 B_1 \frac{\partial B_1}{\partial x} e^{-2B_1^2\bar{\eta}^2}$$

$$- \frac{2}{\sqrt{\pi}}\frac{1}{\sqrt{\pi}}\Omega_1^2 u_1 \frac{\partial u_1}{\partial x} e^{-2B_1^2\bar{\eta}^2} + \frac{2}{\sqrt{\pi}}\frac{1}{\sqrt{\pi}}\Omega_1^2 \frac{u_1^2}{B_1}\frac{\partial B_1}{\partial x} e^{-2B_1^2\bar{\eta}^2} + \left(\frac{2}{\sqrt{\pi}}\right)^2 \Omega_1^2 u_1^2 B_1 \bar{\eta}^2 \frac{\partial B_1}{\partial x} e^{-2B_1^2\bar{\eta}^2} - \left(\frac{2}{\sqrt{\pi}}\right)^2 \Omega_1\Omega_2 u_1^2 B_1 \bar{\eta}^2 \frac{\partial B_2}{\partial x} e^{-(B_1^2+B_2^2)\bar{\eta}^2}$$

$$- \frac{2}{\pi}\Omega_1\Omega_2 u_1 \frac{B_1}{B_2}\frac{\partial u_1}{\partial x} e^{-(B_1^2+B_2^2)\bar{\eta}^2} + \frac{2}{\pi}\Omega_1\Omega_2 u_1^2 \frac{B_1}{B_2^2}\frac{\partial B_2}{\partial x} e^{-(B_1^2+B_2^2)\bar{\eta}^2} + \left(\frac{2}{\sqrt{\pi}}\right)^2 \Omega_1\Omega_2 u_1^2 \bar{\eta}^2 B_1 \frac{\partial B_2}{\partial x} e^{-(B_1^2+B_2^2)\bar{\eta}^2} + \frac{2}{\pi}\Omega_1^2 u_1 \frac{\partial u_1}{\partial x} e^{-B_1^2\bar{\eta}^2}$$

$$- \frac{2}{\pi}\Omega_1^2 \frac{u_1^2}{B_1}\frac{\partial B_1}{\partial x} e^{-B_1^2\bar{\eta}^2} + \frac{2}{\pi}\Omega_1\Omega_2 u_1 \frac{\partial u_1}{\partial x}\frac{B_1}{B_2} e^{-B_1^2\bar{\eta}^2} - \frac{2}{\pi}\Omega_1\Omega_2 u_1^2 \frac{B_1}{B_2^2}\frac{\partial B_2}{\partial x} e^{-B_1^2\bar{\eta}^2} - \frac{2}{\sqrt{\pi}}\Omega_1\Omega_2 u_1 B_2 \bar{\eta}\frac{\partial u_1}{\partial x}\operatorname{erf}(B_1\bar{\eta}) e^{-B_2^2\bar{\eta}^2}$$

$$- \frac{2}{\sqrt{\pi}}\Omega_2^2 u_1 \frac{\partial u_1}{\partial x}\bar{\eta} B_2 \operatorname{erf}(B_2\bar{\eta}) e^{-B_2^2\bar{\eta}^2} - \left(\frac{2}{\sqrt{\pi}}\right)^2 \Omega_1\Omega_2 u_1^2 B_2 \bar{\eta}^2 \frac{\partial B_1}{\partial x} e^{-(B_1^2+B_2^2)\bar{\eta}^2} - \frac{2}{\pi}\Omega_1\Omega_2 u_1 \frac{\partial u_1}{\partial x}\frac{B_2}{B_1} e^{-(B_1^2+B_2^2)\bar{\eta}^2}$$

$$+ \frac{2}{\pi}\Omega_1\Omega_2 u_1^2 \frac{B_2}{B_1^2}\frac{\partial B_1}{\partial x} e^{-(B_1^2+B_2^2)\bar{\eta}^2} + \left(\frac{2}{\sqrt{\pi}}\right)^2 \Omega_1\Omega_2 u_1^2 B_2 \bar{\eta}^2 \frac{\partial B_1}{\partial x} e^{-(B_1^2+B_2^2)\bar{\eta}^2} - \left(\frac{2}{\sqrt{\pi}}\right)^2 \Omega_2^2 u_1^2 B_2 \bar{\eta}^2 \frac{\partial B_2}{\partial x} e^{-2B_2^2\bar{\eta}^2}$$

$$- \frac{2}{\pi}\Omega_2^2 u_1 \frac{\partial u_1}{\partial x} e^{-2B_2^2\bar{\eta}^2} + \frac{2}{\pi} u_1^2 \frac{\Omega_2^2}{B_2}\frac{\partial B_2}{\partial x} e^{-2B_2^2\bar{\eta}^2} + \left(\frac{2}{\sqrt{\pi}}\right)^2 \Omega_2^2 u_1^2 B_2 \bar{\eta}^2 \frac{\partial B_2}{\partial x} e^{-2B_2^2\bar{\eta}^2} + \frac{2}{\pi}\Omega_1\Omega_2 u_1 \frac{\partial u_1}{\partial x}\frac{B_2}{B_1} e^{-B_2^2\bar{\eta}^2}$$

$$- \frac{2}{\pi}\Omega_1\Omega_2 u_1^2 \frac{B_2}{B_1^2}\frac{\partial B_1}{\partial x} e^{-B_2^2\bar{\eta}^2} + \frac{2}{\pi}\Omega_2^2 u_1 \frac{\partial u_1}{\partial x} e^{-B_2^2\bar{\eta}^2} - \frac{2}{\pi}\Omega_2^2 \frac{u_1^2}{B_2}\frac{\partial B_2}{\partial x} e^{-B_2^2\bar{\eta}^2} - \frac{2}{\sqrt{\pi}}\Omega_1 u_1 B_1 \frac{\partial \bar{\eta}}{\partial t} e^{-B_1^2\bar{\eta}^2} - \frac{2}{\sqrt{\pi}}\Omega_2 u_1 B_2 \frac{\partial \bar{\eta}}{\partial t} e^{-B_2^2\bar{\eta}^2}$$

$$- \frac{R\theta_1\delta_1 \operatorname{erf}(A_1\bar{\eta})}{R\theta_1 + \eta P + RT_w}\frac{\partial u_1}{\partial t} - \frac{R\theta_1\delta_2 \operatorname{erf}(A_2\bar{\eta})}{R\theta_1 + \eta P + RT_w}\frac{\partial u_1}{\partial t} - \frac{R\theta_1\delta_1 \operatorname{erf}(A_1\bar{\eta})}{R\theta_1 + \eta P + RT_w} u_1 \frac{\partial u_1}{\partial x} - \frac{\eta P + RT_w}{R\theta_1 + \eta P + RT_w}\frac{\partial u_1}{\partial t} - \frac{R\theta_1\delta_2 \operatorname{erf}(A_2\bar{\eta})}{R\theta_1 + \eta P + RT_w} u_1 \frac{\partial u_1}{\partial x}$$

$$\frac{\rho_1}{\rho}\frac{\partial u_1}{\partial t} + \frac{\rho_1}{\rho} u_1 \frac{\partial u_1}{\partial x} = \frac{R\Theta + \eta P + RT_w}{R\Theta_1 + \eta P + RT_w}\frac{\partial u_1}{\partial t} + \frac{R\Theta+\eta P + RT_w}{R\Theta_1+\eta P+RT_w} u_1 \frac{\partial u_1}{\partial x}$$

$$\frac{\partial}{\partial\bar{y}}\left[(\nu+\epsilon)\frac{\partial}{\partial\bar{y}}\left(\frac{\partial\Psi}{\partial\bar{y}}\right)\right] = \frac{\partial}{\partial\bar y}\left[(\nu+\epsilon)\frac{2}{\sqrt\pi}u_1\left(\Omega_1 B_1 e^{-B_1^2\bar y^2} + \Omega_2 B_2 e^{-B_2^2\bar y^2}\right)\right]$$

Combining all terms of the Momentum Equation and forming the residual, one obtains

$$\Omega_1 \frac{\partial u_1}{\partial t} \operatorname{erf}(B_1\bar y) + \frac{2}{\sqrt\pi}\Omega_1 u_1 \bar y \frac{\partial B_1}{\partial t} e^{-B_1^2\bar y^2} + \frac{2}{\sqrt\pi}\Omega_1 u_1 B_1 \frac{\partial\bar y}{\partial t} e^{-B_1^2\bar y^2} + \Omega_2\frac{\partial u_1}{\partial t}\operatorname{erf}(B_2\bar y \cdots$$

$$+\frac{2}{\sqrt\pi}\Omega_2 u_1 B_2 \frac{\partial \bar y}{\partial t} e^{-B_2^2\bar y^2} + \Omega_1^2 u_1 \frac{\partial u_1}{\partial x}\operatorname{erf}^2(B_1\bar y + \frac{2}{\sqrt\pi}\Omega_1^2 u_1^2 \bar y\frac{\partial B_1}{\partial x}\operatorname{erf}(B_1\bar y)e^{-\cdots}$$

$$+\frac{2}{\sqrt\pi}\Omega_1\Omega_2 u_1^2\bar y\frac{\partial B_2}{\partial x}\operatorname{erf}(B_1\bar y)e^{-B_2^2\bar y^2} + \frac{2}{\sqrt\pi}\Omega_1\Omega_2 u_1^2\bar y\frac{\partial B_1}{\partial x}\operatorname{erf}(B_2\bar y)e^{-B_1^2\bar y^2} + \Omega_2^2 u_1 \cdots$$

$$-\frac{2}{\sqrt\pi}\Omega_1^2 u_1\bar y\frac{\partial u_1}{\partial x}B_1\operatorname{erf}(B_1\bar y)e^{-B_1^2\bar y^2} - \frac{2}{\sqrt\pi}\Omega_1\Omega_2 u_1\bar y\frac{\partial u_1}{\partial x}B_1\operatorname{erf}(B_2\bar y)e^{-B_1^2\bar y^2} - \left(\frac{2}{\sqrt\pi}\right)^2\Omega \cdots$$

$$-\frac{2}{\sqrt\pi}\frac{1}{\sqrt\pi}\Omega_1^2 u_1\frac{\partial u_1}{\partial x}e^{-2B_1^2\bar y^2} + \frac{2}{\sqrt\pi}\frac{1}{\sqrt\pi}\Omega_1^2\frac{u_1^2}{B_1}\frac{\partial B_1}{\partial x}e^{-2B_1^2\bar y^2} + \left(\frac{2}{\sqrt\pi}\right)^2\Omega_1^2 u_1^2 B_1\bar y^2\frac{\partial B_1}{\partial x}e^{-2B_1^2\bar y^2} - \cdots$$

$$-\frac{2}{\pi}\Omega_1\Omega_2 u_1\frac{B_1}{B_2}\frac{\partial u_1}{\partial x}e^{-(B_1^2+B_2^2)\bar y^2} + \frac{2}{\pi}\Omega_1\Omega_2 u_1^2\frac{B_1}{B_2^2}\frac{\partial B_2}{\partial x}e^{-(B_1^2+B_2^2)\bar y^2} + \left(\frac{2}{\sqrt\pi}\right)^2\Omega_1\Omega_2 \cdots$$

$$-\frac{2}{\pi}\Omega_1^2\frac{u_1^2}{B_1}\frac{\partial B_1}{\partial x}e^{-B_1^2\bar y^2} + \frac{2}{\pi}\Omega_1\Omega_2 u_1\frac{\partial u_1}{\partial x}\frac{B_1}{B_2}e^{-B_1^2\bar y^2} - \frac{2}{\pi}\Omega_1\Omega_2 u_1^2\frac{B_1}{B_2^2}\frac{\partial B_2}{\partial x}e^{-B_1^2\bar y^2}$$

$$-\frac{2}{\sqrt\pi}\Omega_2^2 u_1\frac{\partial u_1}{\partial x}\bar y B_2\operatorname{erf}(B_2\bar y)e^{-B_2^2\bar y^2} - \left(\frac{2}{\sqrt\pi}\right)^2\Omega_1\Omega_2 u_1^2 B_2\bar y^2\frac{\partial B_1}{\partial x}e^{-(B_1^2+B_2^2)\bar y^2} - \frac{2}{\pi}\Omega_1\Omega_2 u \cdots$$

$$+\frac{2}{\pi}\Omega_1\Omega_2 u_1^2\frac{B_2}{B_1^2}\frac{\partial B_1}{\partial x}e^{-(B_1^2+B_2^2)\bar y^2} + \left(\frac{2}{\sqrt\pi}\right)^2\Omega_1\Omega_2 u_1^2 B_2\bar y^2\frac{\partial B_1}{\partial x}e^{-(B_1^2+B_2^2)\bar y^2} - \left(\frac{2}{\sqrt\pi}\right)^2\Omega\cdots$$

$$-\frac{2}{\pi}\Omega_2^2 u_1\frac{\partial u_1}{\partial x}e^{-2B_2^2\bar y^2} + \frac{2}{\pi}u_1^2\frac{\Omega_2^2}{B_2}\frac{\partial B_2}{\partial x}e^{-2B_2^2\bar y^2} + \left(\frac{2}{\sqrt\pi}\right)^2\Omega_2^2 u_1^2 B_2\bar y^2\frac{\partial B_2}{\partial x}e^{-2B_2^2\bar y^2} + \cdots$$

$$-\frac{2}{\pi}\Omega_1\Omega_2 u_1^2\frac{B_2}{B_1^2}\frac{\partial B_1}{\partial x}e^{-B_2^2\bar y^2} + \frac{2}{\pi}\Omega_2^2 u_1\frac{\partial u_1}{\partial x}e^{-B_2^2\bar y^2} - \frac{2}{\pi}\Omega_2^2\frac{u_1^2}{B_2}\frac{\partial B_2}{\partial x}e^{-B_2^2\bar y^2} - \frac{2}{\sqrt\pi}\Omega\cdots$$

$$-\frac{R\Theta_1\delta_1\operatorname{erf}(A_1\bar y)}{R\Theta_1+\eta P+RT_w}\frac{\partial u_1}{\partial t} - \frac{R\Theta_1\delta_2\operatorname{erf}(A_2\bar y)}{R\Theta_1+\eta P+RT_w}\frac{\partial u_1}{\partial t} - \frac{R\Theta_1\delta_1\operatorname{erf}(A_1\bar y)}{R\Theta_1+\eta P+RT_w}u_1\frac{\partial u_1}{\partial x} - \frac{\eta P+RT_w}{R\Theta_1+\eta P+R\cdots}$$

$$+\frac{R\theta+\eta P+RT_w}{R\theta_1+\eta P+RT_w}\,u_1\frac{\partial u_1}{\partial x}$$ [3.20]

$$\left(\Omega_1 B_1 e^{-B_1^2\bar{y}^2}+\Omega_2 B_2 e^{-B_2^2\bar{y}^2}\right)\Big]$$ [3.21]

n and forming the residual, one obtains

$$+\frac{2}{\sqrt{\pi}}\Omega_1 u_1 B_1\frac{\partial\bar y}{\partial t}e^{-B_1^2\bar y^2}+\Omega_2\frac{\partial u_1}{\partial t}\,\mathrm{erf}(B_2\bar y)+\frac{2}{\sqrt{\pi}}\Omega_2 u_1\bar y\frac{\partial B_2}{\partial t}e^{-B_2^2\bar y^2}$$

$$f^2(B_1\bar y)+\frac{2}{\sqrt{\pi}}\Omega_1^2u_1^2\bar y\frac{\partial B_1}{\partial x}\mathrm{erf}(B_1\bar y)e^{-B_1^2\bar y^2}+2\Omega_1\Omega_2u_1\frac{\partial u_1}{\partial x}\mathrm{erf}(B_1\bar y)\,\mathrm{erf}(B_2\bar y)$$

$$_1\Omega_2u_1^2\bar y\frac{\partial B_1}{\partial x}\mathrm{erf}(B_2\bar y)e^{-B_1^2\bar y^2}+\Omega_2^2u_1\frac{\partial u_1}{\partial x}\mathrm{erf}^2(B_2\bar y)+\frac{2}{\sqrt{\pi}}\Omega_2^2u_1\bar y\frac{\partial B_2}{\partial x}\mathrm{erf}(B_2\bar y)e^{-B_2^2\bar y^2}$$

$$\Omega_2u_1\bar y\frac{\partial u_1}{\partial x}B_1\mathrm{erf}(B_2\bar y)e^{-B_1^2\bar y^2}-\left(\frac{2}{\sqrt{\pi}}\right)^2\Omega_1^2\Omega_2^2u_1^2\bar y^2B_1\frac{\partial B_1}{\partial x}e^{-2B_1^2\bar y^2}$$

$$e^{-2B_1^2\bar y^2}+\left(\frac{2}{\sqrt{\pi}}\right)^2\Omega_1^2u_1^2B_1\bar y^2\frac{\partial B_1}{\partial x}e^{-2B_1^2\bar y^2}-\left(\frac{2}{\sqrt{\pi}}\right)^2\Omega_1\Omega_2u_1^2B_1\bar y^2\frac{\partial B_2}{\partial x}e^{-(B_1^2+B_2^2)\bar y^2}$$

$$\Omega_2u_1^2\frac{B_1}{B_2^2}\frac{\partial B_2}{\partial x}e^{-(B_1^2+B_2^2)\bar y^2}+\left(\frac{2}{\sqrt{\pi}}\right)^2\Omega_1\Omega_2u_1^2\bar y^2B_1\frac{\partial B_2}{\partial x}e^{-(B_1^2+B_2^2)\bar y^2}+\frac{2}{\pi}\Omega_1^2u_1\frac{\partial u_1}{\partial x}e^{-B_1^2\bar y^2}$$

$$\frac{B_1}{B_2}e^{-B_1^2\bar y^2}-\frac{2}{\pi}\Omega_1\Omega_2u_1^2\frac{B_1}{B_2^2}\frac{\partial B_2}{\partial x}e^{-B_1^2\bar y^2}-\frac{2}{\sqrt{\pi}}\Omega_1\Omega_2u_1B_2\bar y\frac{\partial u_1}{\partial x}\mathrm{erf}(B_1\bar y)e^{-B_2^2\bar y^2}$$

$$\Omega_2u_1^2B_2\bar y^2\frac{\partial B_1}{\partial x}e^{-(B_1^2+B_2^2)\bar y^2}-\frac{2}{\pi}\Omega_1\Omega_2u_1\frac{\partial u_1}{\partial x}\frac{B_2}{B_1}e^{-(B_1^2+B_2^2)\bar y^2}$$

$$\Omega_2u_1^2B_2\bar y^2\frac{\partial B_1}{\partial x}e^{-(B_1^2+B_2^2)\bar y^2}-\left(\frac{2}{\sqrt{\pi}}\right)^2\Omega_2^2u_1^2B_2\bar y^2\frac{\partial B_2}{\partial x}e^{-2B_2^2\bar y^2}$$

$$^{2B_2^2\bar y^2}+\left(\frac{2}{\sqrt{\pi}}\right)^2\Omega_2^2u_1^2B_2\bar y^2\frac{\partial B_2}{\partial x}e^{-2B_2^2\bar y^2}+\frac{2}{\pi}\Omega_1\Omega_2u_1\frac{\partial u_1}{\partial x}\frac{B_2}{B_1}e^{-B_2^2\bar y^2}$$

$$\frac{u_1}{x}e^{-B_2^2\bar y^2}-\frac{2}{\pi}\Omega_2^2\frac{u_1^2}{B_2}\frac{\partial B_2}{\partial x}e^{-B_2^2\bar y^2}-\frac{2}{\sqrt{\pi}}\Omega_1u_1B_1\frac{\partial\bar y}{\partial t}e^{-B_1^2\bar y^2}-\frac{2}{\sqrt{\pi}}\Omega_2u_1B_2\frac{\partial\bar y}{\partial t}e^{-B_2^2\bar y^2}$$

$$-\frac{R\theta_1\delta_1\,\mathrm{erf}(A_1\bar y)}{R\theta_1+\eta P+RT_w}u_1\frac{\partial u_1}{\partial x}-\frac{\eta P+RT_w}{R\theta_1+\eta P+RT_w}\frac{\partial u_1}{\partial t}-\frac{R\theta_1\delta_2\,\mathrm{erf}(A_2\bar y)}{R\theta_1+\eta P+RT_w}u_1\frac{\partial u_1}{\partial x}$$

$$-\frac{\eta P + RT_w}{R\theta_1 + \eta P + RT_w}\,\mu_1 \frac{\partial \mu_1}{\partial x} - \frac{2}{\sqrt{\pi}}\,\mu_1 \frac{\partial}{\partial \bar{y}}\left[(\nu+\epsilon)\left(\Omega_1 B_1 e^{-B_1^2 \bar{y}^2} + \Omega_2 B_2 e^{-B_2^2 \bar{y}^2}\right)\right] = R \qquad [3.22]$$

Multiply Equation 3.22 by the weighting function $\frac{2}{\sqrt{\pi}}\Omega_1 \mu_1 \bar{y} e^{-B_1^2 \bar{y}^2}$ and integrate over $\bar{y}$. The integrals are evaluated analytically. Those integrals not readily obtainable are solved analytically in Appendix A. Solving for $\frac{\partial B_1}{\partial t}$ in Equation 3.22 above, one obtains

$$\begin{aligned}
\frac{\partial B_1}{\partial t} = \Big\{ & -\frac{1}{2}\sqrt{\frac{2}{\pi}}\,\Omega_1^2 \mu_1 \frac{\partial \mu_1}{\partial t}\frac{1}{B_1^2} - \frac{1}{\sqrt{\pi}}\Omega_1 \Omega_2 \mu_1 \frac{\partial \mu_1}{\partial t}\frac{B_2}{B_1^2\sqrt{B_1^2+B_2^2}} - \frac{1}{\sqrt{\pi}}\Omega_1\Omega_2 \mu_1^2 \frac{\partial B_2}{\partial t}\frac{1}{\sqrt{B_1^2+B_2^2}\,(B_1^2+B_2^2)} - \frac{4}{\pi}\frac{1}{\sqrt{2\pi}}\Omega_1^2 \mu_1^2 \frac{\partial \mu_1}{\partial x}\frac{1}{B_1^2}\tan^{-1}(0.707) \\
& - \frac{1}{\pi\sqrt{\pi}}\Omega_1^2 \mu_1^3 \frac{\partial B_1}{\partial x}\left[\frac{1}{\sqrt{2}\,B_1^3}\tan^{-1}(0.707) + \frac{1}{B_1(B_1^2+2B_2^2)}\right] - \frac{4}{\pi\sqrt{\pi}}\Omega_1^2\Omega_2 \mu_1^2 \frac{\partial \mu_1}{\partial x}\left[\frac{1}{\sqrt{2}\,B_1^2}\tan^{-1}\left(\frac{B_2}{\sqrt{2}\,B_1}\right) + \frac{B_2}{B_1^2\sqrt{B_1^2+B_2^2}}\tan^{-1}\left(\frac{B_1}{\sqrt{B_1^2+B_2^2}}\right)\right] \\
& - \frac{2}{\pi}\frac{1}{\sqrt{\pi}}\Omega_1^2\Omega_2 \mu_1^3 \frac{\partial B_2}{\partial x}\left[\frac{1}{(B_1^2+B_2^2)^{3/2}}\tan^{-1}\left(\frac{B_1}{\sqrt{B_1^2+B_2^2}}\right) + \frac{B_1}{(B_1^2+B_2^2)(2B_1^2+B_2^2)}\right] - \frac{2}{\pi}\frac{1}{\sqrt{\pi}}\Omega_1^2\Omega_2 \mu_1^3 \frac{\partial B_1}{\partial x}\left[\frac{1}{(2B_1^2)^{3/2}}\tan^{-1}\left(\frac{B_2}{\sqrt{2}\,B_1}\right) + \frac{B_2}{(2B_1^2)(B_2^2+2B_1^2)}\right] \\
& - \frac{4}{\pi}\frac{1}{\sqrt{\pi}}\Omega_1^2\Omega_2 \mu_1^2 \frac{\partial \mu_1}{\partial x}\frac{B_2}{B_1^2}\tan^{-1}\left(\frac{B_2}{\sqrt{B_1^2+B_2^2}}\right)\frac{1}{\sqrt{B_1^2+B_2^2}} - \frac{2}{\pi}\frac{1}{\sqrt{\pi}}\Omega_1\Omega_2^2 \mu_1^3 \frac{\partial B_2}{\partial x}\left[\frac{1}{(B_1^2+B_2^2)^{3/2}}\tan^{-1}\left(\frac{B_2}{\sqrt{B_1^2+B_2^2}}\right) + \frac{B_2}{(B_1^2+B_2^2)(B_1^2+2B_2^2)}\right] \\
& + \frac{2}{\pi}\frac{1}{\sqrt{\pi}}\Omega_1^3 \mu_1^2 \frac{\partial \mu_1}{\partial x} B_1\left[\frac{1}{(2B_1^2)^{3/2}}\tan^{-1}(0.707) + \frac{1}{6B_1^2}\right] + \frac{2}{\pi}\frac{1}{\sqrt{\pi}}\Omega_1^2\Omega_2 \mu_1^2 \frac{\partial \mu_1}{\partial x} B_1\left[\frac{1}{(2B_1^2)^{3/2}}\tan^{-1}\left(\frac{B_2}{\sqrt{2}\,B_1}\right) + \frac{B_2}{(2B_1^2)(B_2^2+2B_1^2)}\right] + \frac{2}{3}\frac{1}{\pi\sqrt{\pi}}\Omega_1^3 \mu_1^2 \frac{\partial \mu_1}{\partial x}\frac{1}{B_1^2} \\
& - \frac{2}{3}\frac{1}{\pi\sqrt{\pi}}\Omega_1^3 \mu_1^3 \frac{1}{B_1^3}\frac{\partial B_1}{\partial x} + \frac{4}{\pi\sqrt{\pi}}\Omega_1^2\Omega_2 \mu_1^3 B_1 \frac{\partial B_2}{\partial x}\frac{1}{(2B_1^2+B_2^2)^2} + \frac{2}{\pi\sqrt{\pi}}\Omega_1^2\Omega_2 \mu_1^2 \frac{B_1}{B_2}\frac{\partial \mu_1}{\partial x}\frac{1}{(2B_1^2+B_2^2)} - \frac{2}{\pi\sqrt{\pi}}\Omega_1^2\Omega_2 \mu_1^3 \frac{B_1}{B_2^2}\frac{\partial B_2}{\partial x}\frac{1}{(2B_1^2+B_2^2)} \\
& - \frac{4}{\pi\sqrt{\pi}}\Omega_1^2\Omega_2 \mu_1^3 B_1 \frac{\partial B_2}{\partial x}\frac{1}{(2B_1^2+B_2^2)^2} - \frac{1}{\pi\sqrt{\pi}}\Omega_1^3 \frac{\mu_1^2}{B_1^2}\frac{\partial \mu_1}{\partial x} + \frac{1}{\pi\sqrt{\pi}}\Omega_1^3 \frac{\mu_1^3}{B_1^3}\frac{\partial B_1}{\partial x} - \frac{1}{\pi\sqrt{\pi}}\Omega_1^2\Omega_2 \mu_1^2 \frac{\partial \mu_1}{\partial x}\frac{1}{B_1 B_2} + \frac{1}{\pi\sqrt{\pi}}\Omega_1^2\Omega_2 \mu_1^3 \frac{1}{B_1 B_2^2}\frac{\partial B_2}{\partial x} \\
& + \frac{2}{\pi\sqrt{\pi}}\Omega_1^2\Omega_2 \mu_1^2 \frac{\partial \mu_1}{\partial x} B_2\left[\frac{1}{(B_1^2+B_2^2)^{3/2}}\tan^{-1}\left(\frac{B_1}{\sqrt{B_1^2+B_2^2}}\right) + \frac{B_1}{(B_1^2+B_2^2)(2B_1^2+B_2^2)}\right] + \frac{2}{\pi\sqrt{\pi}}\Omega_1\Omega_2^2 \mu_1^2 \frac{\partial \mu_1}{\partial x} B_2\left[\frac{1}{(B_1^2+B_2^2)^{3/2}}\tan^{-1}\left(\frac{B_2}{\sqrt{B_1^2+B_2^2}}\right)\right. \\
& \left. + \frac{B_2}{(B_1^2+B_2^2)(B_1^2+2B_2^2)}\right] + \frac{4}{\pi\sqrt{\pi}}\Omega_1^2\Omega_2 \mu_1^3 B_2 \frac{\partial B_1}{\partial x}\frac{1}{(2B_1^2+B_2^2)^2} + \frac{2}{\pi\sqrt{\pi}}\Omega_1^2\Omega_2 \mu_1^2 \frac{\partial \mu_1}{\partial x}\frac{B_2}{B_1}\frac{1}{(2B_1^2+B_2^2)} - \frac{2}{\pi\sqrt{\pi}}\Omega_1^2\Omega_2 \mu_1^3 \frac{B_2}{B_1^2}\frac{\partial B_1}{\partial x}\frac{1}{(2B_1^2+B_2^2)} \\
& - \frac{4}{\pi\sqrt{\pi}}\Omega_1^2\Omega_2 \mu_1^3 B_2 \frac{\partial B_1}{\partial x}\frac{1}{(2B_1^2+B_2^2)^2} + \frac{4}{\pi\sqrt{\pi}}\Omega_1\Omega_2^2 \mu_1^3 B_2 \frac{\partial B_2}{\partial x}\frac{1}{(B_1^2+2B_2^2)^2} + \frac{2}{\pi\sqrt{\pi}}\Omega_1\Omega_2^2 \mu_1^2 \frac{\partial \mu_1}{\partial x}\frac{1}{(B_1^2+2B_2^2)} \\
& - \frac{2}{\pi\sqrt{\pi}}\Omega_1\Omega_2^2 \frac{\mu_1^3}{B_2}\frac{\partial B_2}{\partial x}\frac{1}{(B_1^2+2B_2^2)} - \frac{4}{\pi\sqrt{\pi}}\Omega_1\Omega_2^2 \mu_1^3 B_2 \frac{\partial B_2}{\partial x}\frac{1}{(B_1^2+2B_2^2)^2} - \frac{2}{\pi\sqrt{\pi}}\Omega_1^2\Omega_2 \mu_1^2 \frac{\partial \mu_1}{\partial x}\frac{B_2}{B_1}\frac{1}{(B_1^2+B_2^2)} + \frac{2}{\pi\sqrt{\pi}}\Omega_1^2\Omega_2 \mu_1^3 \frac{B_2}{B_1^2}\frac{\partial B_1}{\partial x}\frac{1}{(B_1^2+B_2^2)} \\
& - \frac{2}{\pi\sqrt{\pi}}\Omega_1\Omega_2^2 \mu_1^2 \frac{\partial \mu_1}{\partial x}\frac{1}{(B_1^2+B_2^2)} + \frac{2}{\pi\sqrt{\pi}}\Omega_1\Omega_2^2 \frac{\mu_1^3}{B_2}\frac{\partial B_2}{\partial x}\frac{1}{(B_1^2+B_2^2)} + \frac{1}{\sqrt{\pi}}\Omega_1 \delta_1 R\theta_1 \mu_1 \frac{\partial \mu_1}{\partial \theta} A_1 \frac{1}{R\theta_1+\eta P+RT_w}\frac{1}{B_1^2\sqrt{A_1^2+B_1^2}}
\end{aligned}$$

$$-\frac{\eta P + RT_w}{R\theta_1 + \eta P + RT_w}\, u_1 \frac{\partial u_1}{\partial x} - \frac{2}{\sqrt{\pi}}\, u_1 \frac{\partial}{\partial \bar{y}}\left[(\nu+\epsilon)\left(\Omega_1 B_1 e^{-B_1^2\bar{y}^2} + \Omega_2 B_2 e^{-B_2^2\bar{y}^2}\right)\right] = R$$

Multiply Equation 3.22 by the weighting function $\frac{2}{\sqrt{\pi}}\Omega_1 u_1 \bar{y} e^{-B_1^2\bar{y}^2}$ and integrate over…
analytically. Those integrals not readily obtainable are solved analytically in Appe…
Equation 3.22 above, one obtains

$$\frac{\partial B_1}{\partial t} = \Big\{ -\frac{1}{2}\sqrt{\frac{2}{\pi}}\,\Omega_1^2 u_1 \frac{\partial u_1}{\partial t}\frac{1}{B_1^2} - \frac{1}{\sqrt{\pi}}\Omega_1\Omega_2 u_1 \frac{\partial u_1}{\partial t}\frac{B_2}{B_1^2\sqrt{B_1^2+B_2^2}} - \frac{1}{\sqrt{\pi}}\Omega_1\Omega_2 u_1^2 \frac{\partial B_2}{\partial t}\frac{1}{\sqrt{B_1^2+B_2^2}\,(B_1^2 \cdots}$$

$$-\frac{1}{\pi\sqrt{\pi}}\Omega_1^3 u_1^3 \frac{\partial B_1}{\partial x}\left[\frac{1}{\sqrt{2}\,B_1^3}\tan^{-1}(0.707) + \frac{1}{B_1(B_1^2+2B_2^2)}\right] - \frac{4}{\pi\sqrt{\pi}}\Omega_1^2\Omega_2 u_1^2 \frac{\partial u_1}{\partial x}\left[\frac{1}{\sqrt{2}\,B_1^3}\,\mathrm{ta}\cdots\right.$$

$$-\frac{2}{\pi}\frac{1}{\sqrt{\pi}}\Omega_1^2\Omega_2 u_1^3 \frac{\partial B_2}{\partial x}\left[\frac{1}{(B_1^2+B_2^2)^{3/2}}\tan^{-1}\left(\frac{B_1}{\sqrt{B_1^2+B_2^2}}\right) + \frac{B_1}{(B_1^2+B_2^2)(2B_1^2+B_2^2)}\right] - \frac{2}{\pi}\frac{1}{\sqrt{\pi}}\Omega_1^2\Omega_2 u_1 \cdots$$

$$-\frac{4}{\pi}\frac{1}{\sqrt{\pi}}\Omega_1^2\Omega_2 u_1^2\frac{\partial u_1}{\partial x}\frac{B_2}{B_1^2}\tan^{-1}\left(\frac{B_2}{\sqrt{B_1^2+B_2^2}}\right)\frac{1}{\sqrt{B_1^2+B_2^2}} - \frac{2}{\pi}\frac{1}{\sqrt{\pi}}\Omega_1\Omega_2^2 u_1^3\frac{\partial B_2}{\partial x}\left[\frac{1}{(B_1^2+B_2^2)^{3/2}}\cdots\right.$$

$$+\frac{2}{\pi}\frac{1}{\sqrt{\pi}}\Omega_1^3 u_1^2\frac{\partial u_1}{\partial x} B_1\left[\frac{1}{(2B_1^2)^{3/2}}\tan^{-1}(0.707) + \frac{1}{6B_1^2}\right] + \frac{2}{\pi}\frac{1}{\sqrt{\pi}}\Omega_1^2\Omega_2 u_1^2\frac{\partial u_1}{\partial x}B_1\left[\frac{1}{(2B_1^2)^{3/2}}\cdots\right.$$

$$-\frac{2}{3}\frac{1}{\pi\sqrt{\pi}}\Omega_1^3 u_1^3\frac{1}{B_1^3}\frac{\partial B_1}{\partial x} + \frac{4}{\pi\sqrt{\pi}}\Omega_1^2\Omega_2 u_1^3 B_1\frac{\partial B_2}{\partial x}\frac{1}{(2B_1^2+B_2^2)^2} + \frac{2}{\pi\sqrt{\pi}}\Omega_1^2\Omega_2 u_1^2\frac{B_1}{B_2}\frac{\partial u_1}{\partial x}\cdots$$

$$-\frac{4}{\pi\sqrt{\pi}}\Omega_1^2\Omega_2 u_1^3 B_1\frac{\partial B_2}{\partial x}\frac{1}{(2B_1^2+B_2^2)^2} - \frac{1}{\pi\sqrt{\pi}}\Omega_1^3\frac{u_1^2}{B_1^2}\frac{\partial u_1}{\partial x} + \frac{1}{\pi\sqrt{\pi}}\Omega_1^3\frac{u_1^3}{B_1^3}\frac{\partial B_1}{\partial x} - \frac{1}{\pi\sqrt{\pi}}\Omega_1^2\Omega_2\cdots$$

$$+\frac{2}{\pi\sqrt{\pi}}\Omega_1^2\Omega_2 u_1^2\frac{\partial u_1}{\partial x}B_2\left[\frac{1}{(B_1^2+B_2^2)^{3/2}}\tan^{-1}\left(\frac{B_1}{\sqrt{B_1^2+B_2^2}}\right) + \frac{B_1}{(B_1^2+B_2^2)(2B_1^2+B_2^2)}\right] + \frac{2}{\pi\sqrt{\pi}}\Omega_1\Omega_2^2\cdots$$

$$\left. + \frac{B_2}{(B_1^2+B_2^2)(B_1^2+2B_2^2)}\right] + \frac{4}{\pi\sqrt{\pi}}\Omega_1^2\Omega_2 u_1^3 B_2\frac{\partial B_1}{\partial x}\frac{1}{(2B_1^2+B_2^2)^2} + \frac{2}{\pi\sqrt{\pi}}\Omega_1^2\Omega_2 u_1^2\frac{\partial u_1}{\partial x}\frac{B_2}{B_1}\frac{1}{(2B_1^2+B_2^2\cdots}$$

$$-\frac{4}{\pi\sqrt{\pi}}\Omega_1^2\Omega_2 u_1^3 B_2\frac{\partial B_1}{\partial x}\frac{1}{(2B_1^2+B_2^2)^2} + \frac{4}{\pi\sqrt{\pi}}\Omega_1\Omega_2^2 u_1^3 B_2\frac{\partial B_2}{\partial x}\frac{1}{(B_1^2+2B_2^2)^2} + \frac{2}{\pi\sqrt{\pi}}\Omega_1\Omega_2^2\cdots$$

$$-\frac{2}{\pi\sqrt{\pi}}\Omega_1\Omega_2^2\frac{u_1^3}{B_2}\frac{\partial B_2}{\partial x}\frac{1}{(B_1^2+2B_2^2)} - \frac{4}{\pi\sqrt{\pi}}\Omega_1\Omega_2^2 u_1^3 B_2\frac{\partial B_2}{\partial x}\frac{1}{(B_1^2+2B_2^2)^2} - \frac{2}{\pi\sqrt{\pi}}\Omega_1^2\Omega_2 u_1^2\frac{\partial u_1}{\partial x}\cdots$$

$$-\frac{2}{\pi\sqrt{\pi}}\Omega_1\Omega_2^2 u_1^2\frac{\partial u_1}{\partial x}\frac{1}{(B_1^2+B_2^2)} + \frac{2}{\pi\sqrt{\pi}}\Omega_1\Omega_2^2\frac{u_1^3}{B_2}\frac{\partial B_2}{\partial x}\frac{1}{(B_1^2+B_2^2)} + \frac{1}{\sqrt{\pi}}\Omega_1\delta_1 R\theta_1 u_1\frac{\partial}{\partial}\cdots$$

$$\Omega_1 B_1 e^{-B_1^2\bar{y}^2} + \Omega_2 B_2 e^{-B_2^2\bar{y}^2} \Big) \Big] = R \qquad [3.22]$$

ction $\frac{2}{\sqrt{\pi}}\Omega_1 u_1 \bar{y} e^{-B_1^2\bar{y}^2}$ and integrate over $\bar{y}$. The integrals are evaluated
obtainable are solved analytically in Appendix A. Solving for $\frac{\partial B_1}{\partial t}$ in

$$\frac{\partial u_1}{\partial t}\frac{B_2}{B_1^2\sqrt{B_1^2+B_2^2}} - \frac{1}{\sqrt{\pi}}\Omega_1\Omega_2 u_1^2 \frac{\partial B_2}{\partial t}\frac{1}{\sqrt{B_1^2+B_2^2}\,(B_1^2+B_2^2)} - \frac{4}{\pi}\frac{1}{\sqrt{2\pi}}\Omega_1^3 u_1^2\frac{\partial u_1}{\partial x}\frac{1}{B_1^2}\tan^{-1}(0.707)$$

$$+\frac{1}{B_1(B_1^2+2B_2^2)}\Big] - \frac{4}{\pi\sqrt{\pi}}\Omega_1^2\Omega_2 u_1^2\frac{\partial u_1}{\partial x}\Big[\frac{1}{\sqrt{2}\, B_1^2}\tan^{-1}\Big(\frac{B_2}{\sqrt{2}\, B_1}\Big)+\frac{B_2}{B_1^2\sqrt{B_1^2+B_2^2}}\tan^{-1}\Big(\frac{B_1}{\sqrt{B_1^2+B_2^2}}\Big)\Big]$$

$$\Big(\frac{B_1}{\sqrt{B_1^2+B_2^2}}\Big)+\frac{B_1}{(B_1^2+B_2^2)(2B_1^2+B_2^2)}\Big] - \frac{2}{\pi}\frac{1}{\sqrt{\pi}}\Omega_1^2\Omega_2 u_1^3\frac{\partial B_1}{\partial x}\Big[\frac{1}{(2B_1^2)^{3/2}}\tan^{-1}\Big(\frac{B_2}{\sqrt{2}\, B_1}\Big)+\frac{B_2}{(2B_1^2)(B_2^2+2B_1^2)}\Big]$$

$$\Big)\frac{1}{\sqrt{B_1^2+B_2^2}} - \frac{2}{\pi}\frac{1}{\sqrt{\pi}}\Omega_1\Omega_2^2 u_1^3\frac{\partial B_2}{\partial x}\Big[\frac{1}{(B_1^2+B_2^2)^{3/2}}\tan^{-1}\Big(\frac{B_2}{\sqrt{B_1^2+B_2^2}}\Big)+\frac{B_2}{(B_1^2+B_2^2)(B_1^2+2B_2^2)}\Big]$$

$$07) + \frac{1}{6B_1^2}\Big] + \frac{2}{\pi}\frac{1}{\sqrt{\pi}}\Omega_1^2\Omega_2 u_1^2\frac{\partial u_1}{\partial x}B_1\Big[\frac{1}{(2B_1^2)^{3/2}}\tan^{-1}\Big(\frac{B_2}{\sqrt{2}\, B_1}\Big)+\frac{B_2}{(2B_1^2)(B_2^2+2B_1^2)}\Big]+\frac{2}{3}\frac{1}{\pi\sqrt{\pi}}\Omega_1^3 u_1^2\frac{\partial u_1}{\partial x}\frac{1}{B_1^2}$$

$$B_1\frac{\partial B_2}{\partial x}\frac{1}{(2B_1^2+B_2^2)^2} + \frac{2}{\pi\sqrt{\pi}}\Omega_1^2\Omega_2 u_1^2\frac{B_1}{B_2}\frac{\partial u_1}{\partial x}\frac{1}{(2B_1^2+B_2^2)} - \frac{2}{\pi\sqrt{\pi}}\Omega_1^2\Omega_2 u_1^3\frac{B_1}{B_2^2}\frac{\partial B_2}{\partial x}\frac{1}{(2B_1^2+B_2^2)}$$

$$\Omega_1^3\frac{u_1^2}{B_1^2}\frac{\partial u_1}{\partial x}+\frac{1}{\pi\sqrt{\pi}}\Omega_1^3\frac{u_1^3}{B_1^3}\frac{\partial B_1}{\partial x} - \frac{1}{\pi\sqrt{\pi}}\Omega_1^2\Omega_2 u_1^2\frac{\partial u_1}{\partial x}\frac{1}{B_1B_2}+\frac{1}{\pi\sqrt{\pi}}\Omega_1^2\Omega_2 u_1^3\frac{1}{B_1B_2^2}\frac{\partial B_2}{\partial x}$$

$$^{-1}\Big(\frac{B_1}{\sqrt{B_1^2+B_2^2}}\Big)+\frac{B_1}{(B_1^2+B_2^2)(2B_1^2+B_2^2)}\Big] + \frac{2}{\pi\sqrt{\pi}}\Omega_1\Omega_2^2 u_1^2\frac{\partial u_1}{\partial x}B_2\Big[\frac{1}{(B_1^2+B_2^2)^{3/2}}\tan^{-1}\Big(\frac{B_2}{\sqrt{B_1^2+B_2^2}}$$

$$\frac{\partial B_1}{\partial x}\frac{1}{(2B_1^2+B_2^2)^2}+\frac{2}{\pi\sqrt{\pi}}\Omega_1^2\Omega_2 u_1^2\frac{\partial u_1}{\partial x}\frac{B_2}{B_1}\frac{1}{(2B_1^2+B_2^2)} - \frac{2}{\pi\sqrt{\pi}}\Omega_1^2\Omega_2 u_1^3\frac{B_2}{B_1^2}\frac{\partial B_1}{\partial x}\frac{1}{(2B_1^2+B_2^2)}$$

$$\frac{4}{\sqrt{\pi}}\Omega_1\Omega_2^2 u_1^3 B_2\frac{\partial B_2}{\partial x}\frac{1}{(B_1^2+2B_2^2)^2}+\frac{2}{\pi\sqrt{\pi}}\Omega_1\Omega_2^2 u_1^2\frac{\partial u_1}{\partial x}\frac{1}{(B_1^2+2B_2^2)}$$

$$\Omega_2^2 u_1^3 B_2\frac{\partial B_2}{\partial x}\frac{1}{(B_1^2+2B_2^2)^2} - \frac{2}{\pi\sqrt{\pi}}\Omega_1^2\Omega_2 u_1^2\frac{\partial u_1}{\partial x}\frac{B_2}{B_1}\frac{1}{(B_1^2+B_2^2)}+\frac{2}{\pi\sqrt{\pi}}\Omega_1^2\Omega_2 u_1^3\frac{B_2}{B_1^2}\frac{\partial B_1}{\partial x}\frac{1}{(B_1^2+B_2^2)}$$

$$\Omega_1\Omega_2^2\frac{u_1^3}{B_2}\frac{\partial B_2}{\partial x}\frac{1}{(B_1^2+B_2^2)}+\frac{1}{\sqrt{\pi}}\Omega_1\delta_1 R\Theta_1 u_1\frac{\partial u_1}{\partial t}A_1\frac{1}{R\Theta_1+\eta P+RT_w}\frac{1}{B_1^2\sqrt{A_1^2+B_1^2}}$$

$$+\frac{1}{\sqrt{\pi}}\Omega_1\delta_2R\theta_1\mu_1\frac{\partial\mu_1}{\partial t}A_2\frac{1}{R\theta_1+\eta P+RT_w}\frac{1}{B_1^2\sqrt{A_2^2+B_1^2}}+\frac{1}{\sqrt{\pi}}\Omega_1\mu_1\frac{\partial\mu_1}{\partial t}\frac{\eta P+RT_w}{R\theta_1+\eta P+RT_w}\frac{1}{B_1^2}+\frac{1}{\sqrt{\pi}}\Omega_1\delta_1R\theta_1\mu_1^2\frac{\partial\mu_1}{\partial x}A_1\frac{1}{R\theta_1+\eta P+RT_w}\frac{1}{B_1^2\sqrt{A_1^2+B_1^2}}$$

$$+\frac{1}{\sqrt{\pi}}\Omega_1\delta_2R\theta_1\mu_1^2\frac{\partial\mu_1}{\partial x}A_2\frac{1}{R\theta_1+\eta P+RT_w}\frac{1}{B_1^2\sqrt{A_2^2+B_1^2}}+\frac{1}{\sqrt{\pi}}\Omega_1\mu_1^2\frac{\partial\mu_1}{\partial x}\frac{\eta P+RT_w}{R\theta_1+\eta P+RT_w}\frac{1}{B_1^2}$$

$$+\left(\frac{2}{\sqrt{\pi}}\right)^2\Omega_1^2\mu_1^2B_1\left\{\frac{2RC}{P}\left\{\frac{1}{2}\sqrt{\frac{2}{\pi}}A_1^2\theta_1^2\frac{1}{(A_1^2+2B_1^2)\sqrt{A_1^2+B_1^2}}+\frac{1}{\sqrt{\pi}}\theta_1^2A_1A_2\frac{1}{(A_2^2+2B_1^2)\sqrt{A_1^2+A_2^2+2B_1^2}}+\frac{1}{\sqrt{\pi}}\theta_1^2A_1A_2\frac{1}{(A_1^2+2B_1^2)\sqrt{A_1^2+A_2^2+2B_1^2}}\right.\right.$$

$$+\frac{1}{2}\sqrt{\frac{2}{\pi}}\theta_1^2A_2^2\frac{1}{(A_2^2+2B_1^2)\sqrt{A_2^2+B_1^2}}+\frac{1}{\sqrt{\pi}}\theta_1T_wA_1\frac{1}{(A_1^2+2B_1^2)}+\frac{1}{\sqrt{\pi}}\theta_1T_wA_2\frac{1}{(A_2^2+2B_1^2)}\Bigg\}-\frac{2RCB_1^2}{P}\left\{\theta\left[\frac{A_1^2}{16B_1^4}\frac{(2A_1^2+6B_1^2)}{(2B_1^2+A_1^2)^{3/2}}\right.\right.$$

$$-\frac{A_1^3}{3}\left[\frac{A_1(60B_1^4+40A_1B_1^2+8A_1^4)}{64B_1^6(2B_1^2+A_1^2)^{5/2}}\right]+\frac{A_1^5}{10}\left[\frac{A_1}{16(2B_1^2)^4(2B_1^2+A_1^2)^{7/2}}\left[105(2B_1^2)^3+210(2B_1^2)^2A_1^2+168(2B_1^2)A_1^4+48A_1^6\right]\right]$$

$$-\frac{A_1^7}{42}\left[\frac{A_1}{32(2B_1^2)^5(A_1^2+2B_1^2)^{9/2}}\left[945(2B_1^2)^4+2520(2B_1^2)^3A_1^2+3024(2B_1^2)^2A_1^4+1728(2B_1^2)A_1^6+384A_1^8\right]\right]\Bigg]+2\theta_1^2\left[\frac{A_1A_2(2A_2^2+6B_1^2)}{4(2B_1^2)^2(2B_1^2+A_2^2)^{3/2}}\right.$$

$$-\frac{A_1^3}{3}\left[\frac{A_2(15(2B_1^2)^2+20(2B_1^2)A_2+8A_2^4)}{8(2B_1^2)^3(2B_1^2+A_2^2)^{5/2}}\right]+\frac{A_1^5}{10}\left[\frac{A_2}{16(2B_1^2)^4(2B_1^2+A_2^2)^{7/2}}\left[105(2B_1^2)^3+210(2B_1^2)^2A_2^2+168(2B_1^2)A_2^4+48A_2^6\right]\right]$$

$$-\frac{A_1^7}{42}\left[\frac{A_2}{32(2B_1^2)^5(2B_1^2+A_2^2)^{9/2}}\left[945(2B_1^2)^4+2520(2B_1^2)^3A_2^2+3024(2B_1^2)^2A_2^4+1728(2B_1^2)A_2^6+384A_2^8\right]\right]\Bigg]\Bigg]+\theta_1^2\left[\frac{A_2^2(2A_2^2+6B_1^2)}{4(2B_1^2)(2B_1^2+A_2^2)^{3/2}}\right.$$

$$-\frac{A_2^3}{3}\left[\frac{A_2(15(2B_1^2)^2+20(2B_1^2)A_2+8A_2^4)}{8(2B_1^2)^3(2B_1^2+A_2^2)^{5/2}}\right]+\frac{A_2^5}{10}\left[\frac{A_2}{16(2B_1^2)^4(2B_1^2+A_2^2)^{7/2}}\left[105(2B_1^2)^3+210(2B_1^2)^2A_2^2+168(2B_1^2)A_2^4+48A_2^6\right]\right]$$

$$-\frac{A_2^7}{42}\left[\frac{A_2}{32(2B_1^2)^5(2B_1^2+A_2^2)^{9/2}}\left[945(2B_1^2)^4+2520(2B_1^2)^3A_2^2+3024(2B_1^2)^2A_2^4+1728(2B_1^2)A_2^6+384A_2^8\right]\right]\Bigg]\Bigg]+2\theta_1T_w\left[\frac{1}{2\sqrt{\pi}(2B_1^2)^{3/2}}\tan^{-1}\left(\frac{A_1}{\sqrt{2B_1^2}}\right)\right.$$

$$+\left.\frac{A_1}{2\sqrt{\pi}(2B_1^2)(A_1^2+2B_1^2)}\right]+2\theta_1T_w\left[\frac{1}{2\sqrt{\pi}(2B_1^2)^{3/2}}\tan^{-1}\left(\frac{A_2}{\sqrt{2B_1^2}}\right)+\frac{A_2}{2\sqrt{\pi}(2B_1^2)(A_2^2+2B_1^2)}\right]+\frac{\sqrt{2\pi}}{16}\frac{T_w^2}{B_1^3}\Bigg\}+\frac{2}{\sqrt{\pi}}c\eta\theta_1\left\{\frac{A_1}{2(A_1^2+2B_1^2)}+\frac{A_2}{2(A_2^2+2B_1^2)}\right\}$$

$$-2B_1^2c\eta\left\{\theta_1\left[\frac{1}{2\sqrt{\pi}(2B_1^2)^{3/2}}\tan^{-1}\left(\frac{A_1}{\sqrt{2B_1^2}}\right)+\frac{A_1}{2\sqrt{\pi}(2B_1^2)(A_1^2+2B_1^2)}\right]+\theta_1\left[\frac{1}{2\sqrt{\pi}(2B_1^2)^{3/2}}\tan^{-1}\left(\frac{A_2}{\sqrt{2B_1^2}}\right)+\frac{A_2}{2\sqrt{\pi}(2B_1^2)(A_2^2+2B_1^2)}\right]+\frac{\sqrt{2\pi}}{16}\frac{T_w}{B_1^3}\right\}\Bigg\}$$

$$+\left(\frac{2}{\sqrt{\pi}}\right)^2\Omega_1\Omega_2\mu_1^2B_2\left\{\frac{2RC}{P}\left\{\frac{1}{\sqrt{\pi}}\theta_1^2A_1\frac{1}{(A_1^2+B_1^2+B_2^2)\sqrt{2A_1^2+B_1^2+B_2^2}}+\frac{1}{\sqrt{\pi}}\theta_1^2A_1A_2\frac{1}{(A_2^2+B_1^2+B_2^2)\sqrt{A_1^2+A_2^2+B_1^2+B_2^2}}+\frac{1}{\sqrt{\pi}}\theta_1^2A_1A_2\frac{1}{(A_1^2+B_1^2+B_2^2)\sqrt{A_1^2+A_2^2+B_1^2+B_2^2}}\right.\right.$$

$$+\frac{1}{\sqrt{\pi}}\theta_1^2A_2^2\frac{1}{(A_2^2+B_1^2+B_2^2)\sqrt{2A_2^2+B_1^2+B_2^2}}+\frac{1}{\sqrt{\pi}}\theta_1A_1T_w\frac{1}{(A_1^2+B_1^2+B_2^2)}+\frac{1}{\sqrt{\pi}}\theta_1A_2T_w\frac{1}{(A_2^2+B_1^2+B_2^2)}\Bigg\}-2\frac{RCB_2^2}{P}\left\{\theta_1^2\left[\frac{A_1^2(2A_1^2+3(B_1^2+B_2^2))}{4(B_1^2+B_2^2)^2(B_1^2+B_2^2+A_1^2)^{3/2}}\right.\right.$$

$$-\frac{A_1^3}{3}\left[\frac{A_1(15(B_1^2+B_2^2)^2+20(B_1^2+B_2^2)A_1+8A_1^4)}{8(B_1^2+B_2^2)^3(B_1^2+B_2^2+A_1^2)^{5/2}}\right]+\frac{A_1^5}{10}\left[\frac{A_1}{16(B_1^2+B_2^2)^4(B_1^2+B_2^2+A_1^2)^{7/2}}\left[105(B_1^2+B_2^2)^3+210(B_1^2+B_2^2)A_1^2+168(B_1^2+B_2^2)A_1^4+48A_1^6\right]\right]$$

$$+\frac{1}{\sqrt{\pi}}\Omega_1\delta_2 R\theta_1\mu_1\frac{\partial\mu_1}{\partial t}A_2\frac{1}{R\theta_1+\eta P+RT_w}\frac{1}{B_1^2\sqrt{A_2^2+B_1^2}}+\frac{1}{\sqrt{\pi}}\Omega_1\mu_1\frac{\partial\mu_1}{\partial t}\frac{\eta P+RT_w}{R\theta_1+\eta P+RT_w}\frac{1}{B_1^2}+\frac{1}{\sqrt{\pi}}$$

$$+\frac{1}{\sqrt{\pi}}\Omega_1\delta_2 R\theta_1\mu_1^2\frac{\partial\mu_1}{\partial x}A_2\frac{1}{R\theta_1+\eta P+RT_w}\frac{1}{B_1^2\sqrt{A_2^2+B_1^2}}+\frac{1}{\sqrt{\pi}}\Omega_1\mu_1^2\frac{\partial\mu_1}{\partial x}\frac{\eta P+RT_w}{R\theta_1+\eta P+RT_w}\frac{1}{B_1^2}$$

$$+\left(\frac{2}{\sqrt{\pi}}\right)^2\Omega_1^2\mu_1^2 B_1\left\{\frac{2RC}{P}\left\{\frac{1}{2}\sqrt{\frac{2}{\pi}}A_1^2\theta_1^2\frac{1}{(A_1^2+2B_1^2)\sqrt{A_1^2+B_1^2}}+\frac{1}{\sqrt{\pi}}\theta_1^2 A_1A_2\frac{1}{(A_2^2+2B_1^2)\sqrt{A_1^2+A_2^2+}}\right.\right.$$

$$+\frac{1}{2}\sqrt{\frac{2}{\pi}}\theta_1^2A_2^2\frac{1}{(A_2^2+2B_1^2)\sqrt{A_2^2+B_1^2}}+\frac{1}{\sqrt{\pi}}\theta_1T_wA_1\frac{1}{(A_1^2+2B_1^2)}+\frac{1}{\sqrt{\pi}}\theta_1T_wA_2\frac{1}{(A_2^2+2B_1^2)}\Big\}-\frac{2F}{}$$

$$-\frac{A_1^3}{3}\left[\frac{A_1(60B_1^4+40A_1B_1^2+8A_1^4)}{64B_1^6(2B_1^2+A_1^2)^{5/2}}\right]+\frac{A_1^5}{10}\left[\frac{A_1}{16(2B_1^2)^4(2B_1^2+A_1^2)^{7/2}}\left[105(2B_1^2)^3+210(2B_1^2)^2A_1^2+1\right.\right.$$

$$-\frac{A_1^7}{42}\left[\frac{A_1}{32(2B_1^2)^5(A_1^2+2B_1^2)^{9/2}}\left[945(2B_1^2)^4+2520(2B_1^2)^3A_1^2+3024(2B_1^2)^2A_1^4+1728(2B_1^2)A_1^6+384\right.\right.$$

$$-\frac{A_1^3}{3}\left[\frac{A_2(15(2B_1^2)^2+20(2B_1^2)A_2+8A_2^4)}{8(2B_1^2)^3(2B_1^2+A_2^2)^{5/2}}\right]+\frac{A_1^5}{10}\left[\frac{A_2}{16(2B_1^2)^4(2B_1^2+A_2^2)^{7/2}}\left[105(2B_1^2)^3+210(2B\right.\right.$$

$$-\frac{A_1^7}{42}\left[\frac{A_2}{32(2B_1^2)^5(2B_1^2+A_2^2)^{9/2}}\left[945(2B_1^2)^4+2520(2B_1^2)^3A_2^2+3024(2B_1^2)^2A_2^4+1728(2B_1^2)A_2^6\right.\right.$$

$$-\frac{A_2^3}{3}\left[\frac{A_2(15(2B_1^2)^2+20(2B_1^2)A_2+8A_2^4)}{8(2B_1^2)^3(2B_1^2+A_2^2)^{5/2}}\right]+\frac{A_2^5}{10}\left[\frac{A_2}{16(2B_1^2)^4(2B_1^2+A_2^2)^{7/2}}\left[105(2B_1^2)^3+210(2B_1^2)\right.\right.$$

$$-\frac{A_2^7}{42}\left[\frac{A_2}{32(2B_1^2)^5(2B_1^2+A_2^2)^{9/2}}\left[945(2B_1^2)^4+2520(2B_1^2)^3A_2^2+3024(2B_1^2)^2A_2^4+1728(2B_1^2)A_2^6+\right.\right.$$

$$+\frac{A_1}{2\sqrt{\pi}(2B_1^2)(A_1^2+2B_1^2)}\Bigg]+2\theta_1T_w\left[\frac{1}{2\sqrt{\pi}(2B_1^2)^{3/2}}\tan^{-1}\left(\frac{A_2}{\sqrt{2B_1^2}}\right)+\frac{A_2}{2\sqrt{\pi}(2B_1^2)(A_2^2+2B_1^2)}\right]+\frac{\sqrt{2\pi}}{16}\frac{T}{B}$$

$$-2B_1^2C\eta\left\{\theta_1\left[\frac{1}{2\sqrt{\pi}(2B_1^2)^{3/2}}\tan^{-1}\left(\frac{A_1}{\sqrt{2B_1^2}}\right)+\frac{A_1}{2\sqrt{\pi}(2B_1^2)(A_1^2+2B_1^2)}\right]+\theta_1\left[\frac{1}{2\sqrt{\pi}(2B_1^2)^{3/2}}\tan^{-1}\left(\frac{A}{\sqrt{2B}}\right.\right.\right.$$

$$+\left(\frac{2}{\sqrt{\pi}}\right)^2\Omega_1\Omega_2\mu_1^2B_2\left\{\frac{2RC}{P}\left\{\frac{1}{\sqrt{\pi}}\theta_1^2A_1\frac{1}{(A_1^2+B_1^2+B_2^2)\sqrt{2A_1^2+B_1^2+B_2^2}}+\frac{1}{\sqrt{\pi}}\theta_1^2A_1A_2\frac{1}{(A_2^2+B_1^2+B_2^2)\sqrt{A}}\right.\right.$$

$$+\frac{1}{\sqrt{\pi}}\theta_1^2A_2^2\frac{1}{(A_2^2+B_1^2+B_2^2)\sqrt{2A_2^2+B_1^2+B_2^2}}+\frac{1}{\sqrt{\pi}}\theta_1A_1T_w\frac{1}{(A_1^2+B_1^2+B_2^2)}+\frac{1}{\sqrt{\pi}}\theta_1A_2T_w\frac{1}{(A_2^2+B_1^2+B_2^2)}\Big\}-$$

$$-\frac{A_1^3}{3}\left[\frac{A_1(15(B_1^2+B_2^2)^2+20(B_1^2+B_2^2)A_1+8A_1^4)}{8(B_1^2+B_2^2)^3(B_1^2+B_2^2+A_1^2)^{5/2}}\right]+\frac{A_1^5}{10}\left[\frac{A_1}{16(B_1^2+B_2^2)^4(B_1^2+B_2^2+A_1^2)^{7/2}}\left[105(B_1^2+B_2^2)^3\right.\right.$$

$$\cdots \frac{}{B_1} + \frac{1}{\sqrt{\pi}}\Omega_1 \mu_1 \frac{\partial \mu_1}{\partial t}\frac{\eta P + RT_w}{R\theta_1 + \eta P + RT_w}\frac{1}{B_1^2} + \frac{1}{\sqrt{\pi}}\Omega_1 \delta_1 R\theta_1 \mu_1^2 \frac{\partial \mu_1}{\partial x} A_1 \frac{1}{R\theta_1 + \eta P + RT_w}\frac{1}{B_1^2\sqrt{A_1^2+B_1^2}}$$

$$\cdots \frac{}{B_1^2} + \frac{1}{\sqrt{\pi}}\Omega_1 \mu_1^2 \frac{\partial \mu_1}{\partial x}\frac{\eta P + RT_w}{R\theta_1 + \eta P + RT_w}\frac{1}{B_1^2}$$

$$\cdots \frac{1}{\cdots B_1^2)\sqrt{A_1^2+B_1^2}} + \frac{1}{\sqrt{\pi}}\theta_1^2 A_1 A_2 \frac{1}{(A_2^2+2B_1^2)\sqrt{A_1^2+A_2^2+2B_1^2}} + \frac{1}{\sqrt{\pi}}\theta_1^2 A_1 A_2 \frac{1}{(A_1^2+2B_1^2)\sqrt{A_1^2+A_2^2+2B_1^2}}$$

$$\cdots \frac{1}{(A_1^2+2B_1^2)} + \frac{1}{\sqrt{\pi}}\theta_1 T_w A_2 \frac{1}{(A_2^2+2B_1^2)}\Bigg\} - \frac{2RCB_1^2}{P}\Bigg\{\theta\Bigg[\frac{A_1^2}{16 B_1^4}\frac{(2A_1^2+6B_1^2)}{(2B_1^2+A_1^2)^{3/2}}$$

$$\cdots \frac{A_1}{(2B_1^2+A_1^2)^{7/2}}\Big[105(2B_1^2)^3 + 210(2B_1^2)^2A_1^2 + 168(2B_1^2)A_1^4 + 48A_1^6\Big]\Big]$$

$$\cdots (2B_1^2)^3A_1^2 + 3024(2B_1^2)^2A_1^4 + 1728(2B_1^2)A_1^6 + 384A_1^4\Big]\Big]\Big] + 2\theta_1^2\Bigg[\frac{A_1A_2(2A_2^2+6B_1^2)}{4(2B_1^2)^2(2B_1^2+A_2^2)^{3/2}}$$

$$\cdots\Bigg[\frac{A_2}{16(2B_1^2)^4(2B_1^2+A_2^2)^{7/2}}\Big[105(2B_1^2)^3 + 210(2B_1^2)^2A_2^2 + 168(2B_1^2)A_2^4 + 48A_2^6\Big]\Big]$$

$$\cdots 0(2B_1^2)^3A_2^2 + 3024(2B_1^2)^2A_2^4 + 1728(2B_1^2)A_2^6 + 384A_2^4\Big]\Big]\Big] + \theta_1^2\Bigg[\frac{A_2^2(2A_2^2+6B_1^2)}{4(2B_1^2)^2(2B_1^2+A_2^2)^{3/2}}$$

$$\cdots \frac{A_2}{16(2B_1^2)^4(2B_1^2+A_2^2)^{7/2}}\Big[105(2B_1^2)^3 + 210(2B_1^2)^2A_2^2 + 168(2B_1^2)A_2^4 + 48A_2^6\Big]\Big]$$

$$\cdots (2B_1^2)^3A_2^2 + 3024(2B_1^2)^2A_2^4 + 1728(2B_1^2)A_2^6 + 384A_2^4\Big]\Big]\Big] + 2\theta_1 T_w\Bigg[\frac{1}{2\sqrt{\pi}(2B_1^2)^{3/2}}\tan^{-1}\left(\frac{A_1}{\sqrt{2B_1^2}}\right)$$

$$\cdots\tan^{-1}\left(\frac{A_2}{\sqrt{2B_1^2}}\right) + \frac{A_2}{2\sqrt{\pi}(2B_1^2)(A_2^2+2B_1^2)}\Bigg] + \frac{\sqrt{2\pi}}{16}\frac{T_w^2}{B_1^3}\Bigg\} + \frac{2}{\sqrt{\pi}}c\eta\theta_1\Bigg\{\frac{A_1}{2(A_1^2+2B_1^2)} + \frac{A_2}{2(A_2^2+2B_1^2)}\Bigg\}$$

$$\cdots\frac{A_1}{(2B_1^2)(A_1^2+2B_1^2)}\Bigg] + \theta_1\Bigg[\frac{1}{2\sqrt{\pi}(2B_1^2)^{3/2}}\tan^{-1}\left(\frac{A_2}{\sqrt{2B_1^2}}\right) + \frac{A_2}{2\sqrt{\pi}(2B_1^2)(A_2^2+2B_1^2)}\Bigg] + \frac{\sqrt{2\pi}}{16}\frac{T_w}{B_1^3}\Bigg\}\Bigg\}$$

$$\cdots\frac{1}{\cdots)\sqrt{2A_1^2+B_1^2+B_2^2}} + \frac{1}{\sqrt{\pi}}\theta_1^2A_1A_2\frac{1}{(A_2^2+B_1^2+B_2^2)\sqrt{A_1^2+A_2^2+B_1^2+B_2^2}} + \frac{1}{\sqrt{\pi}}\theta_1^2A_1A_2\frac{1}{(A_1^2+B_1^2+B_2^2)\sqrt{A_1^2+A_2^2+B_1^2+B_2^2}}$$

$$\cdots{}_w\frac{1}{(A_1^2+B_1^2+B_2^2)} + \frac{1}{\sqrt{\pi}}\theta_1A_2T_w\frac{1}{(A_2^2+B_1^2+B_2^2)}\Bigg\} - 2\frac{RCB_2^2}{P}\Bigg\{\theta_1^2\Bigg[\frac{A_1^2(2A_1^2+3(B_1^2+B_2^2))}{4(B_1^2+B_2^2)^2(B_1^2+B_2^2+A_1^2)^{3/2}}$$

$$+\frac{A_1^5}{10}\Bigg[\frac{A_1}{16(B_1^2+B_2^2)^4(B_1^2+B_2^2+A_1^2)^{7/2}}\Big[105(B_1^2+B_2^2)^3 + 210(B_1^2+B_2^2)A_1^2 + 168(B_1^2+B_2^2)A_1^4 + 48A_1^6\Big]\Big]$$

$$- \frac{A_1^7}{42}\left[\frac{A_1}{32(B_1^2+B_2^2)^5(B_1^2+B_2^2+A_1^2)^{9/2}}\left[945(B_1^2+B_2^2)^4+2520(B_1^2+B_2^2)^3A_1^2+3024(B_1^2+B_2^2)^2A_1^4+1728(B_1^2+B_2^2)A_1^6+384A_1^8\right]\right]\right]$$

$$+2\theta_1^2\left[\frac{A_1A_2(2A_2^2+3(B_1^2+B_2^2))}{4(B_1^2+B_2^2)^2(B_1^2+B_2^2+A_2^2)^{3/2}}-\frac{A_1^3}{3}\left[\frac{A_2(15(B_1^2+B_2^2)^2+20(B_1^2+B_2^2)A_2+8A_2^4)}{8(B_1^2+B_2^2)^3(B_1^2+B_2^2+A_2^2)^{5/2}}\right]+\frac{A_1^5}{10}\left[\frac{A_2}{16(B_1^2+B_2^2)^4(B_1^2+B_2^2+A_2^2)^{7/2}}\left[105(B_1^2+B_2^2)^3\right.\right.\right.$$

$$\left.\left.+210(B_1^2+B_2^2)^2A_2^2+168(B_1^2+B_2^2)A_2^4+48A_2^6\right]\right]-\frac{A_1^7}{42}\left[\frac{A_1}{32(B_1^2+B_2^2)^5(B_1^2+B_2^2+A_1^2)^{9/2}}\left[945(B_1^2+B_2^2)^4+2520(B_1^2+B_2^2)^3A_1^2\right.\right.$$

$$\left.\left.\left.+3024(B_1^2+B_2^2)^2A_1^4+1728(B_1^2+B_2^2)A_1^6+384A_1^8\right]\right]\right]+\theta_1^2\left[\frac{A_2^2(2A_2^2+3(B_1^2+B_2^2))}{4(B_1^2+B_2^2)^2(B_1^2+B_2^2+A_2^2)^{3/2}}-\frac{A_2^3}{3}\left[\frac{A_2(15(B_1^2+B_2^2)^2+20(B_1^2+B_2^2)A_2+8A_2^4)}{8(B_1^2+B_2^2)^3(B_1^2+B_2^2+A_2^2)^{5/2}}\right]\right.$$

$$\left.+\frac{A_2^5}{10}\left[\frac{A_2}{16(B_1^2+B_2^2)^4(B_1^2+B_2^2+A_2^2)^{7/2}}\left[105(B_1^2+B_2^2)^3+210(B_1^2+B_2^2)^2A_2^2+168(B_1^2+B_2^2)A_2^4+48A_2^6\right]\right]\right]+2\theta_1T_w\left[\frac{1}{2\sqrt{\pi}(B_1^2+B_2^2)^{3/2}}\tan^{-1}\left(\frac{A_1}{\sqrt{B_1^2+B_2^2}}\right)\right.$$

$$\left.+\frac{A_1}{2\sqrt{\pi}(B_1^2+B_2^2)(A_1^2+B_1^2+B_2^2)}\right]+2\theta_1T_w\left[\frac{1}{2\sqrt{\pi}(B_1^2+B_2^2)^{3/2}}\tan^{-1}\left(\frac{A_2}{\sqrt{B_1^2+B_2^2}}\right)+\frac{A_2}{2\sqrt{\pi}(B_1^2+B_2^2)(A_2^2+B_1^2+B_2^2)}\right]+\frac{\sqrt{\pi}}{4}T_w^2\frac{1}{\sqrt{B_1^2+B_2^2}(B_1^2+B_2^2)}\Bigg\}$$

$$+C\eta\left\{\frac{1}{\sqrt{\pi}}\theta_1A_1\frac{1}{(A_1^2+B_1^2+B_2^2)}+\frac{1}{\sqrt{\pi}}\theta_1A_2\frac{1}{(A_2^2+B_1^2+B_2^2)}\right\}-2\eta CB_2^2\left\{\theta_1\left[\frac{1}{2\sqrt{\pi}(B_1^2+B_2^2)^{3/2}}\tan^{-1}\left(\frac{A_1}{\sqrt{B_1^2+B_2^2}}\right)+\frac{A_1}{2\sqrt{\pi}(B_1^2+B_2^2)(A_1^2+B_1^2+B_2^2)}\right]\right.$$

$$\left.+\theta_1\left[\frac{1}{2\sqrt{\pi}(B_1^2+B_2^2)^{3/2}}\tan^{-1}\left(\frac{A_2}{\sqrt{B_1^2+B_2^2}}\right)+\frac{A_2}{2\sqrt{\pi}(B_1^2+B_2^2)(A_2^2+B_1^2+B_2^2)}\right]+\frac{\sqrt{\pi}}{4}T_w\frac{1}{\sqrt{B_1^2+B_2^2}(B_1^2+B_2^2)}\right\}\Bigg\}\Bigg\}\Bigg/\frac{\sqrt{2\pi}}{4\pi}\Omega_1^2\frac{\mu_1^2}{B_1^3} \qquad [3.23]$$

If the residual Equation 3.22 is multiplied by the second weighting function, $\frac{2}{\sqrt{\pi}}\Omega_2\mu_1\xi e^{-B_2^2\xi^2}$, one gets an integral equation that can be handled exactly as was Equation 3.23. Noting again that the integrals are analytically obtained as in the previous case with details of the integrations given in Appendix A and B, one can thus solve for $\frac{\partial B_2}{\partial t}$. This second momentum equation is given by

$$\frac{\partial B_2}{\partial t}=\left\{-\frac{1}{\sqrt{\pi}}\Omega_1\Omega_2\mu_1\frac{\partial\mu_1}{\partial t}\frac{B_1}{B_2^2\sqrt{B_1^2+B_2^2}}-\frac{1}{\sqrt{\pi}}\Omega_1\Omega_2\mu_1^2\frac{\partial B_1}{\partial t}\frac{1}{(B_1^2+B_2^2)\sqrt{B_1^2+B_2^2}}-\frac{1}{2}\sqrt{\frac{2}{\pi}}\Omega_2^2\mu_1\frac{\partial\mu_1}{\partial t}\frac{1}{B_2^2}-\frac{4}{\pi\sqrt{\pi}}\Omega_1^2\Omega_2\mu_1^2\frac{\partial\mu_1}{\partial x}\frac{B_1}{B_2^2}\frac{1}{(B_1^2+B_2^2)^{1/2}}\tan^{-1}\left(\frac{B_1}{(B_1^2+B_2^2)^{1/2}}\right)\right.$$

$$-\frac{2}{\pi\sqrt{\pi}}\Omega_1^2\Omega_2\mu_1^3\frac{\partial B_1}{\partial x}\left[\frac{1}{(B_1^2+B_2^2)^{3/2}}\tan^{-1}\left(\frac{B_1}{(B_1^2+B_2^2)^{1/2}}\right)+\frac{B_1}{(B_1^2+B_2^2)(2B_1^2+B_2^2)}\right]-\frac{4}{\pi\sqrt{\pi}}\Omega_1\Omega_2^2\mu_1^2\frac{\partial\mu_1}{\partial x}\left[\frac{B_1}{B_2^2(B_1^2+B_2^2)^{1/2}}\tan^{-1}\left(\frac{B_2}{(B_1^2+B_2^2)^{1/2}}\right)\right.$$

$$\left.+\frac{1}{\sqrt{2}B_2^2}\tan^{-1}\left(\frac{B_1}{\sqrt{2}B_2}\right)\right]-\frac{2}{\pi\sqrt{\pi}}\Omega_1\Omega_2^2\mu_1^3\frac{\partial B_1}{\partial x}\left[\frac{1}{(2B_2^2)^{3/2}}\tan^{-1}\left(\frac{B_1}{\sqrt{2}B_2}\right)+\frac{B_1}{(2B_2^2)(B_1^2+2B_2^2)}\right]-\frac{2}{\pi\sqrt{\pi}}\Omega_1\Omega_2^2\mu_1^3\frac{\partial B_1}{\partial x}\left[\frac{1}{(B_1^2+B_2^2)^{3/2}}\tan^{-1}\left(\frac{B_2}{(B_1^2+B_2^2)^{1/2}}\right)\right.$$

$$\left.+\frac{B_2}{(B_1^2+B_2^2)(B_1^2+2B_2^2)}\right]-\frac{4}{\pi\sqrt{2\pi}}\Omega_2^3\mu_1^2\frac{\partial\mu_1}{\partial x}\frac{1}{B_2^2}\tan^{-1}(0.707)-\frac{2}{\pi\sqrt{\pi}}\Omega_2^3\mu_1^3\frac{\partial B_2}{\partial x}\left[\frac{1}{(2B_2^2)^{3/2}}\tan^{-1}(0.707)+\frac{1}{6B_2^2}\right]$$

$$+\frac{2}{\pi\sqrt{\pi}}\Omega_1^2\Omega_2\mu_1^3\frac{\partial\mu_1}{\partial x}\left[\frac{1}{(B_1^2+B_2^2)^{3/2}}\tan^{-1}\left(\frac{B_1}{(B_1^2+B_2^2)^{1/2}}\right)+\frac{B_1}{(B_1^2+B_2^2)}\right]+\frac{2}{\pi\sqrt{\pi}}\Omega_1\Omega_2^2\mu_1^2\frac{\partial\mu_1}{\partial x}B_1\left[\frac{1}{(B_1^2+B_2^2)^{3/2}}\tan^{-1}\left(\frac{B_2}{(B_1^2+B_2^2)^{1/2}}\right)\right.$$

$$- \frac{A_1^7}{42}\left[\frac{A_1}{32(B_1^2+B_2^2)^5(B_1^2+B_2^2+A_1^2)^{9/2}}\left[945(B_1^2+B_2^2)^4+2520(B_1^2+B_2^2)^3A_1^2+3024(B_1^2+B_2^2)^2A_1^4+\right.\right.$$

$$+ 2\theta_1^2\left[\frac{A_1A_2(2A_2^2+3(B_1^2+B_2^2))}{4(B_1^2+B_2^2)^2(B_1^2+B_2^2+A_2^2)^{3/2}} - \frac{A_1^3}{3}\left[\frac{A_2(15(B_1^2+B_2^2)^2+20(B_1^2+B_2^2)A_2+8A_2^4)}{8(B_1^2+B_2^2)^3(B_1^2+B_2^2+A_2^2)^{5/2}}\right] + \frac{A_1^5}{10}\left[\frac{}{16\cdots}\right.\right.$$

$$+ 210(B_1^2+B_2^2)^2A_2^2+168(B_1^2+B_2^2)A_2^4+48A_2^6\Big]\Big] - \frac{A_1^7}{42}\left[\frac{A_1}{32(B_1^2+B_2^2)^5(B_1^2+B_2^2+A_1^2)^{9/2}}\right[\cdots$$

$$+ 3024(B_1^2+B_2^2)^2A_1^4+1728(B_1^2+B_2^2)A_1^6+384A_1^4\Big]\Big]\Big] + \theta_1^2\left[\frac{A_2^2(2A_2^2+3(B_1^2+B_2^2))}{4(B_1^2+B_2^2)^2(B_1^2+B_2^2+A_2^2)^{3/2}} - \cdots\right.$$

$$+ \frac{A_2^5}{10}\left[\frac{A_2}{16(B_1^2+B_2^2)^4(B_1^2+B_2^2+A_2^2)^{7/2}}\left[105(B_1^2+B_2^2)^3+210(B_1^2+B_2^2)^2A_2^2+168(B_1^2+B_2^2)A_2^4+48A_2^6\cdots\right.\right.$$

$$+ \frac{A_1}{2\sqrt{\pi}(B_1^2+B_2^2)(A_1^2+B_1^2+B_2^2)}\Big] + 2\theta_1T_w\left[\frac{1}{2\sqrt{\pi}(B_1^2+B_2^2)^{3/2}}\tan^{-1}\left(\frac{A_2}{\sqrt{B_1^2+B_2^2}}\right) + \frac{A_2}{2\sqrt{\pi}(B_1^2+B_2^2)(A_2^2+B_1^2\cdots}\right.$$

$$+ C\eta\left\{\frac{1}{\sqrt{\pi}}\theta_1A_1\frac{1}{(A_1^2+B_1^2+B_2^2)} + \frac{1}{\sqrt{\pi}}\theta_1A_2\frac{1}{(A_2^2+B_1^2+B_2^2)}\right\} - 2\eta C B_2^2\left\{\theta_1\left[\frac{1}{2\sqrt{\pi}(B_1^2+B_2^2)^{3/2}}\tan\cdots\right.\right.$$

$$+ \theta_1\left[\frac{1}{2\sqrt{\pi}(B_1^2+B_2^2)^{3/2}}\tan^{-1}\left(\frac{A_2}{\sqrt{B_1^2+B_2^2}}\right) + \frac{A_2}{2\sqrt{\pi}(B_1^2+B_2^2)(A_2^2+B_1^2+B_2^2)}\right] + \frac{\sqrt{\pi}}{4}T_w\frac{1}{\sqrt{B_1^2+B_2^2}(B_1^2+\cdots}$$

If the residual Equation 3.22 is multiplied by the second weighting function, $\frac{2}{\sqrt{\pi}}$
equation that can be handled exactly as was Equation 3.23. Noting again that the
as in the previous case with details of the integrations given in Appendix A and
This second momentum equation is given by

$$\frac{\partial B_2}{\partial t} = \left\{-\frac{1}{\sqrt{\pi}}\Omega_1\Omega_2\mu_1\frac{\partial\mu_1}{\partial t}\frac{B_1}{B_2^2\sqrt{B_1^2+B_2^2}} - \frac{1}{\sqrt{\pi}}\Omega_1\Omega_2\mu_1^2\frac{\partial B_1}{\partial t}\frac{1}{(B_1^2+B_2^2)\sqrt{B_1^2+B_2^2}} - \frac{1}{2}\sqrt{\frac{2}{\pi}}\Omega_2^2\mu_1\frac{\partial\mu_1}{\partial t}\cdots\right.$$

$$- \frac{2}{\pi\sqrt{\pi}}\Omega_1^2\Omega_2\mu_1^3\frac{\partial B_1}{\partial x}\left[\frac{1}{(B_1^2+B_2^2)^{3/2}}\tan^{-1}\left(\frac{B_1}{(B_1^2+B_2^2)^{1/2}}\right) + \frac{B_1}{(B_1^2+B_2^2)(2B_1^2+B_2^2)}\right] - \frac{4}{\pi\sqrt{\pi}}\Omega_1\cdots$$

$$+ \frac{1}{\sqrt{2}B_2^2}\tan^{-1}\left(\frac{B_1}{\sqrt{2}B_2}\right)\Big] - \frac{2}{\pi\sqrt{\pi}}\Omega_1\Omega_2^2\mu_1^3\frac{\partial B_1}{\partial x}\left[\frac{1}{(2B_2^2)^{3/2}}\tan^{-1}\left(\frac{B_1}{\sqrt{2}B_2}\right) + \frac{B_1}{(2B_2^2)(B_1^2+2B_2^2)}\right]$$

$$+ \frac{B_2}{(B_1^2+B_2^2)(B_1^2+2B_2^2)}\Big] - \frac{4}{\pi\sqrt{2\pi}}\Omega_2^3\mu_1^2\frac{\partial\mu_1}{\partial x}\frac{1}{B_2^2}\tan^{-1}(0.707) - \frac{2}{\pi\sqrt{\pi}}\Omega_2^3\mu_1^3\frac{\partial B_2}{\partial x}\left[\frac{1}{(2B_2^2)^{3/\cdots}}\right.$$

$$+ \frac{2}{\pi\sqrt{\pi}}\Omega_1^2\Omega_2\mu_1^2\frac{\partial\mu_1}{\partial x}\left[\frac{1}{(B_1^2+B_2^2)^{3/2}}\tan^{-1}\left(\frac{B_1}{(B_1^2+B_2^2)^{1/2}}\right) + \frac{B_1}{(B_1^2+B_2^2)}\right] + \frac{2}{\pi\sqrt{\pi}}\Omega_1\Omega_2^2\mu_1^2\frac{\partial\cdots}{\partial\cdots}$$

$$)^4 + 2520(B_1^2+B_2^2)^3 A_1^2 + 3024(B_1^2+B_2^2)^2 A_1^4 + 1728(B_1^2+B_2^2) A_1^6 + 384 A_1^4 \Big] \Big] \Big]$$

$$(B_1^2+B_2^2)^2 + 20(B_1^2+B_2^2)A_2 + 8A_2^4) \over B_1^2+B_2^2)^3(B_1^2+B_2^2+A_2^2)^{5/2} \Big] + \frac{A_1^5}{10}\Big[\frac{A_2}{16(B_1^2+B_2^2)^4(B_1^2+B_2^2+A_2^2)^{7/2}}\Big[105(B_1^2+B_2^2)^3$$

$$\Big]\Big] - \frac{A_1^7}{42}\Big[\frac{A_1}{32(B_1^2+B_2^2)^5(B_1^2+B_2^2+A_1^2)^{9/2}}\Big[945(B_1^2+B_2^2)^4 + 2520(B_1^2+B_2^2)^3 A_1^2$$

$$A_1^4\Big]\Big]\Big] + \theta_1^2\Big[\frac{A_2^2(2A_2^2+3(B_1^2+B_2^2))}{4(B_1^2+B_2^2)^2(B_1^2+B_2^2+A_2^2)^{3/2}} - \frac{A_2^3}{3}\Big[\frac{A_2(15(B_1^2+B_2^2)^2+20(B_1^2+B_2^2)A_2+8A_2^4)}{8(B_1^2+B_2^2)^3(B_1^2+B_2^2+A_2^2)^{5/2}}\Big]$$

$$)^3 + 210(B_1^2+B_2^2)^2 A_2^2 + 168(B_1^2+B_2^2)A_2^4 + 48A_2^6\Big]\Big]\Big] + 2\theta_1 T_w\Big[\frac{1}{2\sqrt{\pi}(B_1^2+B_2^2)^{3/2}}\tan^{-1}\Big(\frac{A_1}{\sqrt{B_1^2+B_2^2}}\Big)$$

$$+B_2^2)^{3/2}}\tan^{-1}\Big(\frac{A_2}{\sqrt{B_1^2+B_2^2}}\Big) + \frac{A_2}{2\sqrt{\pi}(B_1^2+B_2^2)(A_2^2+B_1^2+B_2^2)}\Big] + \frac{\sqrt{\pi}}{4}T_w^2\frac{1}{\sqrt{B_1^2+B_2^2}(B_1^2+B_2^2)}\Big\}$$

$$+B_2^2)}\Big\} - 2\eta C B_2^2\Big\{\Theta_1\Big[\frac{1}{2\sqrt{\pi}(B_1^2+B_2^2)^{3/2}}\tan^{-1}\Big(\frac{A_1}{\sqrt{B_1^2+B_2^2}}\Big) + \frac{A_1}{2\sqrt{\pi}(B_1^2+B_2^2)(A_1^2+B_1^2+B_2^2)}\Big]$$

$$\frac{A_2}{B_1^2+B_2^2)(A_2^2+B_1^2+B_2^2)}\Big] + \frac{\sqrt{\pi}}{4}T_w\frac{1}{\sqrt{B_1^2+B_2^2}(B_1^2+B_2^2)}\Big\}\Big\}\Big\}\Big/\frac{\sqrt{2\pi}}{4\pi}\Omega_1^2\frac{\mu_1^2}{B_1^3} \qquad [3.23]$$

ed by the second weighting function, $\frac{2}{\sqrt{\pi}}\Omega_2\mu_1\bar{\eta}e^{-B_2^2\bar{\eta}^2}$, one gets an integral
as Equation 3.23. Noting again that the integrals are analytically obtained
he integrations given in Appendix A and B, one can thus solve for $\frac{\partial B_2}{\partial t}$

$$\mu_1^2\frac{\partial B_1}{\partial t}\frac{1}{(B_1^2+B_2^2)\sqrt{B_1^2+B_2^2}} - \frac{1}{2}\sqrt{\frac{2}{\pi}}\Omega_2^2\mu_1\frac{\partial\mu_1}{\partial t}\frac{1}{B_2^2} - \frac{4}{\pi\sqrt{\pi}}\Omega_1^2\Omega_2\mu_1^2\frac{\partial\mu_1}{\partial x}\frac{B_1}{B_2^2}\frac{1}{(B_1^2+B_2^2)^{1/2}}\tan^{-1}\Big(\frac{B_1}{(B_1^2+B_2^2)^{1/2}}\Big)$$

$$\Big(\frac{B_1}{(B_1^2+B_2^2)^{1/2}}\Big) + \frac{B_1}{(B_1^2+B_2^2)(2B_1^2+B_2^2)}\Big] - \frac{4}{\pi\sqrt{\pi}}\Omega_1\Omega_2^2\mu_1^2\frac{\partial\mu_1}{\partial x}\Big[\frac{B_1}{B_2^2(B_1^2+B_2^2)^{1/2}}\tan^{-1}\Big(\frac{B_2}{(B_1^2+B_2^2)^{1/2}}\Big)$$

$$\frac{\partial B_1}{\partial x}\Big[\frac{1}{(2B_2^2)^{3/2}}\tan^{-1}\Big(\frac{B_1}{\sqrt{2}B_2}\Big) + \frac{B_1}{(2B_2^2)(B_1^2+2B_2^2)}\Big] - \frac{2}{\pi\sqrt{\pi}}\Omega_1\Omega_2^2\mu_1^3\frac{\partial B_1}{\partial x}\Big[\frac{1}{(B_1^2+B_2^2)^{3/2}}\tan^{-1}\Big(\frac{B_2}{(B_1^2+B_2^2)^{1/2}}\Big)$$

$$\frac{1}{B_2^2}\tan^{-1}(0.707) - \frac{2}{\pi\sqrt{\pi}}\Omega_2^3\mu_1^3\frac{\partial B_2}{\partial x}\Big[\frac{1}{(2B_2^2)^{3/2}}\tan^{-1}(0.707) + \frac{1}{6B_2^3}\Big]$$

$$\frac{B_1}{B_1^2+B_2^2)^{1/2}}\Big) + \frac{B_1}{(B_1^2+B_2^2)}\Big] + \frac{2}{\pi\sqrt{\pi}}\Omega_1\Omega_2^2\mu_1^2\frac{\partial\mu_1}{\partial x}B_1\Big[\frac{1}{(B_1^2+B_2^2)^{3/2}}\tan^{-1}\Big(\frac{B_2}{(B_1^2+B_2^2)^{1/2}}\Big)$$

$$+\frac{B_2}{(B_1^2+B_2^2)(B_1^2+2B_2^2)}\Big] + \frac{4}{\pi\sqrt{\pi}}\,\Omega_1^2\Omega_2^3\mu_1^3 B_1\frac{\partial B_1}{\partial x}\frac{1}{(2B_1^2+B_2^2)^2} + \frac{2}{\pi\sqrt{\pi}}\,\Omega_1^2\Omega_2\mu_1^2\frac{\partial \mu_1}{\partial x}\frac{1}{(2B_1^2+B_2^2)} - 2\frac{\Omega_1^2\Omega_2\mu_1^3}{\pi\sqrt{\pi}\;B_1}\frac{\partial B_1}{\partial x}\frac{1}{(2B_1^2+B_2^2)}$$

$$-\frac{4}{\pi\sqrt{\pi}}\,\Omega_1^2\Omega_2\mu_1^3 B_1\frac{\partial B_1}{\partial x}\frac{1}{(2B_1^2+B_2^2)^2} + \frac{4}{\pi\sqrt{\pi}}\,\Omega_1\Omega_2^2\mu_1^3\frac{\partial B_2}{\partial x}B_2\frac{1}{(B_1^2+2B_2^2)^2} + \frac{2}{\pi\sqrt{\pi}}\,\Omega_1\Omega_2^2\mu_1^2\frac{\partial \mu_1}{\partial x}\frac{B_1}{B_2}\frac{1}{(B_1^2+2B_2^2)}$$

$$-\frac{2}{\pi\sqrt{\pi}}\,\Omega_1\Omega_2^2\mu_1^3\frac{B_1}{B_2^2}\frac{\partial B_2}{\partial x}\frac{1}{(B_1^2+2B_2^2)} - \frac{4}{\pi\sqrt{\pi}}\,\Omega_1\Omega_2^2\mu_1^3 B_1\frac{\partial B_2}{\partial x}\frac{1}{(B_1^2+2B_2^2)^2} - \frac{2}{\pi\sqrt{\pi}}\,\Omega_1^2\Omega_2\mu_1^2\frac{\partial \mu_1}{\partial x}\frac{1}{(B_1^2+B_2^2)} + \frac{2}{\pi\sqrt{\pi}}\,\Omega_1^2\Omega_2\frac{\mu_1^3}{B_1}\frac{\partial B_1}{\partial x}\frac{1}{(B_1^2+B_2^2)}$$

$$-\frac{2}{\pi\sqrt{\pi}}\,\Omega_1\Omega_2^2\mu_1^2\frac{\partial \mu_1}{\partial x}\frac{B_1}{B_2}\frac{1}{(B_1^2+B_2^2)} + \frac{2}{\pi\sqrt{\pi}}\,\Omega_1\Omega_2^2\mu_1^3\frac{B_1}{B_2^2}\frac{\partial B_2}{\partial x}\frac{1}{(B_1^2+B_2^2)} + \frac{2}{\pi\sqrt{\pi}}\,\Omega_1\Omega_2^2\mu_1^2\frac{\partial \mu_1}{\partial x}B_2\Big[\frac{1}{(2B_2^2)^{3/2}}\tan^{-1}\Big(\frac{B_1}{\sqrt{2}\,B_2}\Big) + \frac{B_1}{2B_2^2(B_1^2+2B_2^2)}\Big]$$

$$+\frac{2}{\pi\sqrt{\pi}}\,\Omega_2^3\mu_1^2\frac{\partial \mu_1}{\partial x}B_2\Big[\frac{1}{(2B_2^2)^{3/2}}\tan^{-1}(0.707) + \frac{1}{2B_2(3B_2^2)}\Big] + \frac{4}{\pi\sqrt{\pi}}\,\Omega_1\Omega_2^2\mu_1^3 B_2\frac{\partial B_1}{\partial x}\frac{1}{(B_1^2+2B_2^2)^2} + \frac{2}{\pi\sqrt{\pi}}\,\Omega_1\Omega_2^2\mu_1^2\frac{\partial \mu_1}{\partial x}\frac{B_2}{B_1}\frac{1}{(B_1^2+2B_2^2)}$$

$$-\frac{2}{\pi\sqrt{\pi}}\,\Omega_1\Omega_2^2\mu_1^3\frac{B_2}{B_1^2}\frac{\partial B_1}{\partial x}\frac{1}{(B_1^2+2B_2^2)} - \frac{2}{\pi\sqrt{\pi}}\,\Omega_1\Omega_2^2\mu_1^3 B_2\frac{\partial B_1}{\partial x}\frac{1}{(B_1^2+2B_2^2)^2} + \frac{4}{9\pi\sqrt{\pi}}\,\Omega_2^3\mu_1^3\frac{\partial B_2}{\partial x}\frac{1}{B_2^3} + \frac{2}{3\pi\sqrt{\pi}}\,\Omega_2^3\mu_1^2\frac{\partial \mu_1}{\partial x}\frac{1}{B_2^2}$$

$$-\frac{2}{\pi\sqrt{\pi}}\,\Omega_2^3\mu_1^3\frac{\partial B_2}{\partial x}\frac{1}{B_2^3} - \frac{4}{9\pi\sqrt{\pi}}\,\Omega_2^3\mu_1^3\frac{\partial B_2}{\partial x}\frac{1}{B_2^3} - \frac{1}{\pi\sqrt{\pi}}\,\Omega_1\Omega_2^2\mu_1^2\frac{\partial \mu_1}{\partial x}\frac{1}{B_1B_2} + \frac{1}{\pi\sqrt{\pi}}\,\Omega_1\Omega_2^2\mu_1^3\frac{\partial B_1}{\partial x}\frac{1}{B_1^2B_2} - \frac{1}{\pi\sqrt{\pi}}\,\Omega_2^3\mu_1^2\frac{\partial \mu_1}{\partial x}\frac{1}{B_2^2}$$

$$+\frac{4}{\pi\sqrt{\pi}}\,\Omega_2^3\mu_1^3\frac{\partial B_2}{\partial x}\frac{1}{B_2^3} + \frac{1}{\sqrt{\pi}}\,\delta_1\Omega_2 R\theta_1 A_1\mu_1\frac{\partial \mu_1}{\partial t}\frac{1}{R\theta_1+\eta P+RT_w}\frac{1}{B_2^2(A_1^2+B_2^2)^{1/2}} + \frac{1}{\sqrt{\pi}}\,\delta_2\Omega_2 R\theta_1 A_2\mu_1\frac{\partial \mu_1}{\partial t}\frac{1}{R\theta_1+\eta P+RT_w}\frac{1}{B_2^2(A_2^2+B_2^2)^{1/2}}$$

$$+\frac{1}{\sqrt{\pi}}\,\Omega_2\mu_1\frac{\partial \mu_1}{\partial t}\frac{\eta P+RT_w}{R\theta_1+\eta P+RT_w}\frac{1}{B_2^2} + \frac{1}{\sqrt{\pi}}\,\delta_1\Omega_2 R\theta_1 A_1\mu_1^2\frac{\partial \mu_1}{\partial x}\frac{1}{R\theta_1+\eta P+RT_w}\frac{1}{B_2^2(A_1^2+B_2^2)^{1/2}} + \frac{1}{\sqrt{\pi}}\,\delta_2\Omega_2 R\theta_1 A_2\mu_1^2\frac{\partial \mu_1}{\partial x}\frac{1}{R\theta_1+\eta P+RT_w}\frac{1}{B_2^2(A_2^2+B_2^2)^{1/2}}$$

$$+\frac{1}{\sqrt{\pi}}\,\Omega_2\mu_1^2\frac{\partial \mu_1}{\partial x}\frac{\eta P+RT_w}{R\theta_1+\eta P+RT_w}\frac{1}{B_2^2} + \frac{4}{\pi}\,\Omega_1\Omega_2\mu_1^2 B_1\Big\{\frac{2}{\sqrt{\pi}}\frac{RC}{P}\Big[\theta_1^2A_1^2\frac{1}{(A_1^2+B_1^2+B_2^2)}\frac{1}{(2A_1^2+B_1^2+B_2^2)^{1/2}} + \theta_1^2A_2^2\frac{1}{(A_2^2+B_1^2+B_2^2)}\frac{1}{(2A_2^2+B_1^2+B_2^2)^{1/2}}$$

$$+\theta_1^2A_1A_2\frac{1}{(A_1^2+B_1^2+B_2^2)}\frac{1}{(A_1^2+A_2^2+B_1^2+B_2^2)^{1/2}} + \theta_1^2A_2^2\frac{1}{(A_2^2+B_1^2+B_2^2)}\frac{1}{(2A_2^2+B_1^2+B_2^2)^{1/2}} + \theta_1T_wA_1\frac{1}{(A_1^2+B_1^2+B_2^2)} + \theta_1T_wA_2\frac{1}{(A_2^2+B_1^2+B_2^2)}\Big]$$

$$-2\frac{RCB_1^2}{P}\Big\{\theta_1^2\Big[\frac{A_1^2(2A_1^2+3(B_1^2+B_2^2))}{4(B_1^2+B_2^2)^2(B_1^2+B_2^2+A_1^2)^{3/2}} - \frac{A_1^3}{3}\Big[\frac{A_1(15(B_1^2+B_2^2)^2+20(B_1^2+B_2^2)A_1+8A_1^4}{8(B_1^2+B_2^2)^3(B_1^2+B_2^2+A_1^2)^{5/2}}\Big] + \frac{A_1^5}{10}\frac{A_1}{16(B_1^2+B_2^2)^4(B_1^2+B_2^2+A_1^2)^{7/2}}\Big[105(B_1^2+B_2^2)^3$$

$$+210(B_1^2+B_2^2)^2A_1^2 + 168(B_1^2+B_2^2)A_1^4 + 48A_1^6\Big] - \frac{A_1^7}{42}\frac{A_1}{32(B_1^2+B_2^2)^5(B_1^2+B_2^2+A_1^2)^{9/2}}\Big[945(B_1^2+B_2^2)^4 + 2520(B_1^2+B_2^2)^3A_1^2 + 3024(B_1^2+B_2^2)^2A_1^4$$

$$+1728(B_1^2+B_2^2)A_1^6 + 384A_1^8\Big]\Big] + 2\theta_1^2\Big[\frac{A_1A_2(2A_2^2+3(B_1^2+B_2^2))}{4(B_1^2+B_2^2)^2(B_1^2+B_2^2+A_2^2)^{3/2}} - \frac{A_1^3}{3}\Big[\frac{A_2(15(B_1^2+B_2^2)^2+20(B_1^2+B_2^2)A_2+8A_2^4)}{8(B_1^2+B_2^2)^3(B_1^2+B_2^2+A_2^2)^{5/2}}\Big]$$

$$+\frac{A_1^5}{10}\frac{A_2}{16(B_1^2+B_2^2)^4(B_1^2+B_2^2+A_2^2)^{7/2}}\Big[105(B_1^2+B_2^2)^3 + 210(B_1^2+B_2^2)^2A_2^2 + 168(B_1^2+B_2^2)A_2^4 + 48A_2^6\Big] - \frac{A_1^7}{42}\frac{A_2}{32(B_1^2+B_2^2)^5(B_1^2+B_2^2+A_2^2)^{9/2}}\Big[945(B_1^2+B_2^2)^4$$

$$+\frac{B_2}{(B_1^2+B_2^2)(B_1^2+2B_2^2)}\Big] + \frac{4}{\pi\sqrt{\pi}}\Omega_1^2\Omega_2^3\mu_1^3 B_1\frac{\partial B_1}{\partial x}\frac{1}{(2B_1^2+B_2^2)^2} + \frac{2}{\pi\sqrt{\pi}}\Omega_1^2\Omega_2\mu_1^2\frac{\partial\mu_1}{\partial x}\frac{1}{(2B_1^2+B_2^2)} - 2$$

$$-\frac{4}{\pi\sqrt{\pi}}\Omega_1^2\Omega_2\mu_1^3 B_1\frac{\partial B_1}{\partial x}\frac{1}{(2B_1^2+B_2^2)^2} + \frac{4}{\pi\sqrt{\pi}}\Omega_1\Omega_2^2\mu_1^3\frac{\partial B_2}{\partial x}B_2\frac{1}{(B_1^2+2B_2^2)^2} + \frac{2}{\pi\sqrt{\pi}}\Omega_1\Omega_2^2\mu$$

$$-\frac{2}{\pi\sqrt{\pi}}\Omega_1\Omega_2^2\mu_1^3\frac{B_1}{B_2^2}\frac{\partial B_2}{\partial x}\frac{1}{(B_1^2+2B_2^2)} - \frac{4}{\pi\sqrt{\pi}}\Omega_1\Omega_2^2\mu_1^3 B_1\frac{\partial B_2}{\partial x}\frac{1}{(B_1^2+2B_2^2)^2} - \frac{2}{\pi\sqrt{\pi}}\Omega_1^2\Omega_2\mu_1^2\,\partial$$

$$-\frac{2}{\pi\sqrt{\pi}}\Omega_1\Omega_2^2\mu_1^2\frac{\partial\mu_1}{\partial x}\frac{B_1}{B_2}\frac{1}{(B_1^2+B_2^2)} + \frac{2}{\pi\sqrt{\pi}}\Omega_1\Omega_2^2\mu_1^3\frac{B_1}{B_2^2}\frac{\partial B_2}{\partial x}\frac{1}{(B_1^2+B_2^2)} + \frac{2}{\pi\sqrt{\pi}}\Omega_1\Omega_2^2\mu_1^2\frac{\partial\mu_1}{\partial x}B_2$$

$$+\frac{2}{\pi\sqrt{\pi}}\Omega_2^3\mu_1^2\frac{\partial\mu_1}{\partial x}B_2\left[\frac{1}{(2B_2^2)^{3/2}}\tan^{-1}(0.707) + \frac{1}{2B_2(3B_2^2)}\right] + \frac{4}{\pi\sqrt{\pi}}\Omega_1\Omega_2^2\mu_1^3 B_2\frac{\partial B_1}{\partial x}\frac{1}{(B_1^2+2B_2^2)}$$

$$-\frac{2}{\pi\sqrt{\pi}}\Omega_1\Omega_2^2\mu_1^3\frac{B_2}{B_1^2}\frac{\partial B_1}{\partial x}\frac{1}{(B_1^2+2B_2^2)} - \frac{2}{\pi\sqrt{\pi}}\Omega_1\Omega_2^2\mu_1^3 B_2\frac{\partial B_1}{\partial x}\frac{1}{(B_1^2+2B_2^2)^2} + \frac{4}{9\pi\sqrt{\pi}}\Omega_2^3\mu_1^3\frac{\partial B_2}{\partial x}$$

$$-\frac{2}{\pi\sqrt{\pi}}\Omega_2^3\mu_1^3\frac{\partial B_2}{\partial x}\frac{1}{B_2^3} - \frac{4}{9\pi\sqrt{\pi}}\Omega_2^3\mu_1^3\frac{\partial B_2}{\partial x}\frac{1}{B_2^3} - \frac{1}{\pi\sqrt{\pi}}\Omega_1\Omega_2^2\mu_1^2\frac{\partial\mu_1}{\partial x}\frac{1}{B_1B_2} + \frac{1}{\pi\sqrt{\pi}}\Omega_1\Omega_2^2\mu_1^3\frac{\partial B_1}{\partial x}$$

$$+\frac{4}{\pi\sqrt{\pi}}\Omega_2^3\mu_1^3\frac{\partial B_2}{\partial x}\frac{1}{B_2^3} + \frac{1}{\sqrt{\pi}}\delta_1\Omega_2 R\Theta_1 A_1\mu_1\frac{\partial\mu_1}{\partial t}\frac{1}{R\Theta_1+\eta P+RT_w}\frac{1}{B_2^2(A_1^2+B_2^2)^{1/2}} + \frac{1}{\sqrt{\pi}}\delta_2\Omega_2 R\Theta_1 A$$

$$+\frac{1}{\sqrt{\pi}}\Omega_2\mu_1\frac{\partial\mu_1}{\partial t}\frac{\eta P+RT_w}{R\Theta_1+\eta P+RT_w}\frac{1}{B_2^2} + \frac{1}{\sqrt{\pi}}\delta_1\Omega_2 R\Theta_1 A_1\mu_1^2\frac{\partial\mu_1}{\partial x}\frac{1}{R\Theta_1+\eta P+RT_w}\frac{1}{B_2^2(A_1^2+B_2^2)^{1/2}} + \frac{1}{\sqrt{\pi}}$$

$$+\frac{1}{\sqrt{\pi}}\Omega_2\mu_1^2\frac{\partial\mu_1}{\partial x}\frac{\eta P+RT_w}{R\Theta_1+\eta P+RT_w}\frac{1}{B_2^2} + \frac{4}{\pi}\Omega_1\Omega_2\mu_1^2 B_1\left\{\frac{2}{\sqrt{\pi}}\frac{RC}{P}\left[\Theta_1^2A_1^2\frac{1}{(A_1^2+B_1^2+B_2^2)}\frac{1}{(2A_1^2+B_1^2+B_2^2)^{1/2}}\right.\right.$$

$$+\Theta_1^2A_1A_2\frac{1}{(A_1^2+B_1^2+B_2^2)}\frac{1}{(A_1^2+A_2^2+B_1^2+B_2^2)^{1/2}} + \Theta_1^2A_2^2\frac{1}{(A_2^2+B_1^2+B_2^2)}\frac{1}{(2A_2^2+B_1^2+B_2^2)^{1/2}} + \Theta_1T_wA_1\frac{}{(A_1^2+B}$$

$$-2\frac{RCB_1^2}{P}\left\{\Theta_1^2\left[\frac{A_1^2(2A_1^2+3(B_1^2+B_2^2))}{4(B_1^2+B_2^2)^2(B_1^2+B_2^2+A_1^2)^{3/2}} - \frac{A_1^3}{3}\left[\frac{A_1(15(B_1^2+B_2^2)^2+20(B_1^2+B_2^2)A_1+8A_1^4}{8(B_1^2+B_2^2)^3(B_1^2+B_2^2+A_1^2)^{5/2}}\right] + \frac{A_1^5}{10}\,\frac{}{16}\right.\right.$$

$$+210(B_1^2+B_2^2)^2A_1^2 + 168(B_1^2+B_2^2)A_1^4 + 48A_1^6\Big] - \frac{A_1^7}{42}\frac{A_1}{32(B_1^2+B_2^2)^5(B_1^2+B_2^2+A_1^2)^{9/2}}\Big[945(B_1^2+B_2^2)^4+$$

$$+1728(B_1^2+B_2^2)A_1^6 + 384A_1^4\Big]\Big] + 2\Theta_1^2\left[\frac{A_1A_2(2A_2^2+3(B_1^2+B_2^2))}{4(B_1^2+B_2^2)^2(B_1^2+B_2^2+A_2^2)^{3/2}} - \frac{A_1^3}{3}\left[\frac{A_2(15(B_1^2+B_2^2)^2+2}{8(B_1^2+B_2^2)^3(B_1^2+}\right.\right.$$

$$+\frac{A_1^5}{10}\frac{A_2}{16(B_1^2+B_2^2)^4(B_1^2+B_2^2+A_2^2)^{7/2}}\Big[105(B_1^2+B_2^2)^3 + 210(B_1^2+B_2^2)^2A_2^2 + 168(B_1^2+B_2^2)A_2^4 + 48A_2^6\Big] -$$

$$\frac{\partial B_1}{\partial x}\frac{1}{(2B_1^2+B_2^2)^2} + \frac{2}{\pi\sqrt{\pi}}\Omega_1^2\Omega_2\mu_1^2\frac{\partial \mu_1}{\partial x}\frac{1}{(2B_1^2+B_2^2)} - 2\frac{\Omega_1^2\Omega_2\mu_1^3}{\pi\sqrt{\pi}\ B_1}\frac{\partial B_1}{\partial x}\frac{1}{(2B_1^2+B_2^2)}$$

$$\frac{4}{\pi\sqrt{\pi}}\Omega_1\Omega_2^2\mu_1^3\frac{\partial B_2}{\partial x}B_2\frac{1}{(B_1^2+2B_2^2)^2} + \frac{2}{\pi\sqrt{\pi}}\Omega_1\Omega_2^2\mu_1^2\frac{\partial \mu_1}{\partial x}\frac{B_1}{B_2}\frac{1}{(B_1^2+2B_2^2)}$$

$$\frac{4}{\sqrt{\pi}}\Omega_1\Omega_2^2\mu_1^3 B_1\frac{\partial B_2}{\partial x}\frac{1}{(B_1^2+2B_2^2)^2} - \frac{2}{\pi\sqrt{\pi}}\Omega_1^2\Omega_2\mu_1^2\frac{\partial \mu_1}{\partial x}\frac{1}{(B_1^2+B_2^2)} + \frac{2}{\pi\sqrt{\pi}}\Omega_1^2\Omega_2\frac{\mu_1^3}{B_1}\frac{\partial B_1}{\partial x}\frac{1}{(B_1^2+B_2^2)}$$

$$\Omega_1\Omega_2^2\mu_1^3\frac{B_1}{B_2^2}\frac{\partial B_2}{\partial x}\frac{1}{(B_1^2+B_2^2)} + \frac{2}{\pi\sqrt{\pi}}\Omega_1\Omega_2^2\mu_1^2\frac{\partial \mu_1}{\partial x}B_2\left[\frac{1}{(2B_2^2)^{3/2}}\tan^{-1}\left(\frac{B_1}{\sqrt{2}\,B_2}\right) + \frac{B_1}{2B_2^2(B_1^2+2B_2^2)}\right]$$

$$7) + \frac{1}{2B_2(3B_2^2)}\Bigg] + \frac{4}{\pi\sqrt{\pi}}\Omega_1\Omega_2^2\mu_1^3 B_2\frac{\partial B_1}{\partial x}\frac{1}{(B_1^2+2B_2^2)^2} + \frac{2}{\pi\sqrt{\pi}}\Omega_1\Omega_2^2\mu_1^2\frac{\partial \mu_1}{\partial x}\frac{B_2}{B_1}\frac{1}{(B_1^2+2B_2^2)}$$

$$\frac{}{\sqrt{\pi}}\Omega_1\Omega_2^2\mu_1^3 B_2\frac{\partial B_1}{\partial x}\frac{1}{(B_1^2+2B_2^2)^2} + \frac{4}{9\pi\sqrt{\pi}}\Omega_2^3\mu_1^3\frac{\partial B_2}{\partial x}\frac{1}{B_2^3} + \frac{2}{3\pi\sqrt{\pi}}\Omega_2^3\mu_1^2\frac{\partial \mu_1}{\partial x}\frac{1}{B_2^2}$$

$$\frac{1}{B_2^3} - \frac{1}{\pi\sqrt{\pi}}\Omega_1\Omega_2^2\mu_1^2\frac{\partial \mu_1}{\partial x}\frac{1}{B_1B_2} + \frac{1}{\pi\sqrt{\pi}}\Omega_1\Omega_2^2\mu_1^3\frac{\partial B_1}{\partial x}\frac{1}{B_1^2B_2} - \frac{1}{\pi\sqrt{\pi}}\Omega_2^3\mu_1^2\frac{\partial \mu_1}{\partial x}\frac{1}{B_2^2}$$

$$\mu_1\frac{\partial \mu_1}{\partial t}\frac{1}{R\Theta_1+\eta P+RT_w}\frac{1}{B_2^2(A_1^2+B_2^2)^{1/2}} + \frac{1}{\sqrt{\pi}}\delta_2\Omega_2 R\Theta_1 A_2\mu_1\frac{\partial \mu_1}{\partial t}\frac{1}{R\Theta_1+\eta P+RT_w}\frac{1}{B_2^2(A_2^2+B_2^2)^{1/2}}$$

$$\delta_1\Omega_2 R\Theta_1 A_1\mu_1^2\frac{\partial \mu_1}{\partial x}\frac{1}{R\Theta_1+\eta P+RT_w}\frac{1}{B_2^2(A_1^2+B_2^2)^{1/2}} + \frac{1}{\sqrt{\pi}}\delta_2\Omega_2 R\Theta_1 A_2\mu_1^2\frac{\partial \mu_1}{\partial x}\frac{1}{R\Theta_1+\eta P+RT_w}\frac{1}{B_2^2(A_2^2+B_2^2)^{1/2}}$$

$$\Omega_2\mu_1^2 B_1\left\{\frac{2}{\sqrt{\pi}}\frac{RC}{P}\left[\Theta_1^2A_1^2\frac{1}{(A_1^2+B_1^2+B_2^2)}\frac{1}{(2A_1^2+B_1^2+B_2^2)^{1/2}} + \Theta_1^2A_2^2\frac{1}{(A_2^2+B_1^2+B_2^2)}\frac{1}{(2A_2^2+B_1^2+B_2^2)^{1/2}}\right.\right.$$

$$\left. + \Theta_1^2A_2^2\frac{1}{(A_2^2+B_1^2+B_2^2)}\frac{1}{(2A_2^2+B_1^2+B_2^2)^{1/2}} + \Theta_1T_wA_1\frac{1}{(A_1^2+B_1^2+B_2^2)} + \Theta_1T_wA_2\frac{1}{(A_2^2+B_1^2+B_2^2)}\right]$$

$$\frac{A_1^3}{3}\left[\frac{A_1(15(B_1^2+B_2^2)^2+20(B_1^2+B_2^2)A_1+8A_1^4}{8(B_1^2+B_2^2)^3(B_1^2+B_2^2+A_1^2)^{5/2}}\right] + \frac{A_1^5}{10}\frac{A_1}{16(B_1^2+B_2^2)^4(B_1^2+B_2^2+A_1^2)^{7/2}}\Big[105(B_1^2+B_2^2)^3$$

$$A_1^6\Big] - \frac{A_1^7}{42}\frac{A_1}{32(B_1^2+B_2^2)^5(B_1^2+B_2^2+A_1^2)^{9/2}}\Big[945(B_1^2+B_2^2)^4+2520(B_1^2+B_2^2)^3A_1^2+3024(B_1^2+B_2^2)^2A_1^4$$

$$\frac{A_2(2A_2^2+3(B_1^2+B_2^2))}{(B_1^2+B_2^2)^2(B_1^2+B_2^2+A_2^2)^{3/2}} - \frac{A_1^3}{3}\left[\frac{A_2(15(B_1^2+B_2^2)^2+20(B_1^2+B_2^2)A_2+8A_2^4)}{8(B_1^2+B_2^2)^3(B_1^2+B_2^2+A_2^2)^{5/2}}\right]$$

$$B_2^2)^3+210(B_1^2+B_2^2)^2A_2^2+168(B_1^2+B_2^2)A_2^4+48A_2^6\Big] - \frac{A_1^7}{42}\frac{A_2}{32(B_1^2+B_2^2)^5(B_1^2+B_2^2+A_2^2)^{9/2}}\Big[945(B_1^2+B_2^2)^4$$

$$+2520(B_1^2+B_2^2)^3A_2^2+3024(B_1^2+B_2^2)^2A_2^4+1728(B_1^2+B_2^2)A_2^6+384A_2^8\Big]\Big]+\theta_1^2\left[\frac{A_2^3(2A_2^2+3(B_1^2+B_2^2))}{4(B_1^2+B_2^2)^2(B_1^2+B_2^2+A_2^2)^{3/2}}\right.$$

$$-\frac{A_2^3}{3}\left[\frac{A_2(15(B_1^2+B_2^2)^2+20(B_1^2+B_2^2)A_2+8A_2^4)}{8(B_1^2+B_2^2)^3(B_1^2+B_2^2+A_2^2)^{5/2}}\right]+\frac{A_2^5}{10}\frac{A_2}{16(B_1^2+B_2^2)^4(B_1^2+B_2^2+A_2^2)^{7/2}}\Big[105(B_1^2+B_2^2)^3+210(B_1^2+B_2^2)^2A_2^2+168(B_1^2+B_2^2)A_2^4$$

$$+48A_2^6\Big]-\frac{A_2^7}{42}\frac{A_2}{32(B_1^2+B_2^2)^5(B_1^2+B_2^2+A_2^2)^{9/2}}\Big[945(B_1^2+B_2^2)^4+2520(B_1^2+B_2^2)^3A_2^2+3024(B_1^2+B_2^2)^2A_2^4+1728(B_1^2+B_2^2)A_2^6+384A_2^8\Big]\Big]$$

$$+\theta_1T_w\left[\frac{1}{(B_1^2+B_2^2)^{3/2}}\tan^{-1}\left(\frac{A_1}{(B_1^2+B_2^2)^{1/2}}\right)+\frac{A_1}{(B_1^2+B_2^2)(A_1^2+B_1^2+B_2^2)}\right]+\frac{\theta_1T_w}{\sqrt{\pi}}\left[\frac{1}{(B_1^2+B_2^2)^{3/2}}\tan^{-1}\left(\frac{A_2}{(B_1^2+B_2^2)^{1/2}}\right)+\frac{A_2}{(B_1^2+B_2^2)(A_2^2+B_1^2+B_2^2)}\right]$$

$$+\frac{\sqrt{\pi}}{4}T_w^2\frac{1}{(B_1^2+B_2^2)^{1/2}(B_1^2+B_2^2)}\Big\}+\frac{1}{\sqrt{\pi}}C\eta\theta_1A_1\frac{1}{(A_1^2+B_1^2+B_2^2)}+\frac{1}{\sqrt{\pi}}C\eta\theta_1A_2\frac{1}{(A_2^2+B_1^2+B_2^2)}-2\eta CB_1^2\left[\frac{\theta_1}{2\sqrt{\pi}}\left[\frac{1}{(B_1^2+B_2^2)^{3/2}}\tan^{-1}\left(\frac{A_1}{(B_1^2+B_2^2)^{1/2}}\right)\right.\right.$$

$$+\left.\frac{A_1}{(B_1^2+B_2^2)(A_1^2+B_1^2+B_2^2)}\right]+\frac{\theta_1}{2\sqrt{\pi}}\left[\frac{1}{(B_1^2+B_2^2)^{3/2}}\tan^{-1}\left(\frac{A_2}{(B_1^2+B_2^2)^{1/2}}\right)+\frac{A_2}{(B_1^2+B_2^2)(A_2^2+B_1^2+B_2^2)}\right]+T_w\frac{\sqrt{\pi}}{4(B_1^2+B_2^2)^{1/2}(B_1^2+B_2^2)}\Big]\Big\}$$

$$+\left(\frac{2}{\sqrt{\pi}}\right)^2\Omega_2^2\mu_1^2B_2\left\{\frac{2RC}{P}\left[\frac{\sqrt{2}\,\theta_1A_1^2}{(A_1^2+2B_2^2)(A_1^2+B_2^2)^{1/2}}+\frac{\theta_1^2A_1A_2}{\sqrt{\pi}(A_1^2+2B_2^2)(A_1^2+A_2^2+2B_2^2)^{1/2}}+\frac{\theta_1^2A_1A_2}{(A_1^2+2B_2^2)(A_1^2+A_2^2+2B_2^2)^{1/2}}+\frac{\sqrt{2}\,\theta_1^2A_2^2}{(A_2^2+2B_2^2)(A_2^2+B_2^2)^{1/2}}\right.\right.$$

$$+\left.\frac{\theta_1A_1T_w}{(A_1^2+2B_2^2)}+\frac{\theta_1A_2T_w}{(A_2^2+2B_2^2)}\right]-\frac{2RCB_2^2}{P}\left[\theta_1^2\left[\frac{A_1^3(2A_1^2+6B_2^2)}{16B_2^4(2B_2^2+A_1^2)^{3/2}}-\frac{A_1^3}{3}\frac{A_1(60B_2^4+40A_1B_2^2+8A_1^4)}{64B_2^6(2B_2^2+A_1^2)^{5/2}}\right.\right.$$

$$+\frac{A_1^5}{10}\frac{A_2}{16(B_1^2+B_2^2)^4(B_1^2+B_2^2+A_2^2)^{7/2}}\Big[105(B_1^2+B_2^2)^3+210(B_1^2+B_2^2)^2A_2^2+168(B_1^2+B_2^2)A_2^4+48A_2^6\Big]-\frac{A_2^7}{42}\frac{A_2}{32(B_1^2+B_2^2)^5(B_1^2+B_2^2+A_2^2)^{9/2}}\Big[945(B_1^2+B_2^2)^4$$

$$+2520(B_1^2+B_2^2)^3A_2^2+3024(B_1^2+B_2^2)^2A_2^4+1728(B_1^2+B_2^2)A_2^6+384A_2^8\Big]\Big]+2\theta_1^2\left[\frac{A_1A_2(2A_2^2+6B_2^2)}{4(2B_2^2)^2(2B_2^2+A_2^2)^{3/2}}-\frac{A_1^3}{3}\frac{A_2(15(2B_2^2)^2+20(2B_2)A_2+8A_2^4)}{8(2B_2^2)^3(2B_2^2+A_2^2)^{5/2}}\right.$$

$$+\frac{A_1^5}{10}\frac{A_2}{16(2B_2^2)^4(2B_2^2+A_2^2)^{7/2}}\Big[105(2B_2^2)^3+210(2B_2^2)^2A_2^2+168(2B_2^2)A_2^4+48A_2^6\Big]-\frac{A_1^7}{42}\frac{A_2}{32(2B_2^2)^5(2B_2^2+A_2^2)^{9/2}}\Big[945(2B_2^2)^4+2520(2B_2^2)^3A_2^2$$

$$+3024(2B_2^2)^2A_2^4+1728(2B_2^2)A_2^6+384A_2^8\Big]\Big]+\theta_1^2\left[\frac{A_2^3(2A_2^2+6B_2^2)}{16B_2^4(2B_2^2+A_2^2)^{3/2}}-\frac{A_2^3}{3}\frac{A_2(60B_2^4+40A_2B_2^2+8A_2^4)}{64B_2^6(2B_2^2+A_2^2)^{5/2}}\right.$$

$$+\frac{A_2^5}{10}\frac{A_2}{16(2B_2^2)^4(2B_2^2+A_2^2)^{7/2}}\Big[105(2B_2^2)^3+210(2B_2^2)^2A_2^2+168(2B_2^2)A_2^4+48A_2^6\Big]-\frac{A_2^7}{42}\frac{A_2}{32(2B_2^2)^5(A_2^2+2B_2^2)^{9/2}}\Big[945(2B_2^2)^4+2520(2B_2^2)^3A_2^2$$

$$+3024(2B_2^2)^2A_2^4+1728(2B_2^2)A_2^6+384A_2^8\Big]\Big]+\frac{\theta_1T_w}{\sqrt{\pi}}\left[\frac{1}{(2B_2^2)^{3/2}}\tan^{-1}\left(\frac{A_1}{\sqrt{2}B_2}\right)+\frac{A_1}{(2B_2^2)(B_1^2+2B_2^2)}\right]+\frac{\theta_1T_w}{\sqrt{\pi}}\left[\frac{1}{(2B_2^2)^{3/2}}\tan^{-1}\left(\frac{A_2}{\sqrt{2}B_2}\right)\right.$$

$$+\left.\frac{A_2}{(2B_2^2)(B_1^2+2B_2^2)}\right]+\frac{\sqrt{2\pi}}{16}\frac{T_w^2}{B_2^3}\Big]+\frac{C\eta A_1A_2}{\sqrt{\pi}(A_1^2+2B_2^2)}+\frac{C\eta\theta_1A_2}{\sqrt{\pi}(A_2^2+2B_2^2)}-2C\eta B_2^2\left[\frac{\theta_1}{2\sqrt{\pi}}\left[\frac{1}{(2B_2^2)^{3/2}}\tan^{-1}\left(\frac{A_1}{\sqrt{2}B_2}\right)+\frac{A_1}{(2B_2^2)(B_1^2+2B_2^2)}\right]\right.$$

$$+ 2520(B_1^2+B_2^2)^3 A_2^2 + 3024(B_1^2+B_2^2)^2 A_2^4 + 1728(B_1^2+B_2^2) A_2^6 + 384 A_2^8 \Big] \Big] + \theta_1^2 \Big[\frac{A_2^2(2A_2^2 + 3(\ldots}{4(B_1^2+B_2^2)^2(B_1 \ldots}$$

$$- \frac{A_2^3}{3} \Big[\frac{A_2(15(B_1^2+B_2^2)^2 + 20(B_1^2+B_2^2)A_2 + 8A_2^4}{8(B_1^2+B_2^2)^3(B_1^2+B_2^2+A_2^2)^{5/2}} \Big] + \frac{A_2^5}{10} \frac{A_2}{16(B_1^2+B_2^2)^4(B_1^2+B_2^2+A_2^2)^{7/2}} \Big[105(B_1^2+ \ldots$$

$$+ 48 A_2^6 \Big] - \frac{A_2^7}{42} \frac{A_2}{32(B_1^2+B_2^2)^5(B_1^2+B_2^2+A_2^2)^{9/2}} \Big[945(B_1^2+B_2^2)^4 + 2520(B_1^2+B_2^2)^3 A_2^2 + 3024(\ldots$$

$$+ \theta_1 T_w \Big[\frac{1}{(B_1^2+B_2^2)^{3/2}} \tan^{-1}\Big(\frac{A_1}{(B_1^2+B_2^2)^{1/2}} \Big) + \frac{A_1}{(B_1^2+B_2^2)(A_1^2+B_1^2+B_2^2)} \Big] + \frac{\theta_1 T_w}{\sqrt{\pi}} \Big[\frac{1}{(B_1^2+B_2^2)^{3/2}} \tan^{-1}\Big(\ldots$$

$$+ \frac{\sqrt{\pi}}{4} T_w^2 \frac{1}{(B_1^2+B_2^2)^{1/2}(B_1^2+B_2^2)} \Big\} + \frac{1}{\sqrt{\pi}} C \eta \theta_1 A_1 \frac{1}{(A_1^2+B_1^2+B_2^2)} + \frac{1}{\sqrt{\pi}} C \eta \theta_1 A_2 \frac{1}{(A_2^2+B_1^2+B_2^2)} - 2 \eta C B_1^2 \ldots$$

$$+ \frac{A_1}{(B_1^2+B_2^2)(A_1^2+B_1^2+B_2^2)} \Big] + \frac{\theta_1}{2\sqrt{\pi}} \Big[\frac{1}{(B_1^2+B_2^2)^{3/2}} \tan^{-1}\Big(\frac{A_2}{(B_1^2+B_2^2)^{1/2}} \Big) + \frac{A_2}{(B_1^2+B_2^2)(A_2^2+B_1^2+B_2^2)} \Big] + T_w - \ldots$$

$$+ \Big(\frac{2}{\sqrt{\pi}} \Big)^2 \Omega_2^2 \mu_1^2 B_2 \Big\{ \frac{2RC}{P} \Big[\frac{\sqrt{2}\,\theta_1 A_1^2}{(A_1^2+2B_2^2)(A_1^2+B_2^2)^{1/2}} + \frac{\theta_1^2 A_1 A_2}{\sqrt{\pi}(A_2^2+2B_2^2)(A_1^2+A_2^2+2B_2^2)^{1/2}} + \frac{\theta_1^2 A_1 \ldots}{(A_1^2+2B_2^2)(A_1^2+A \ldots}$$

$$+ \frac{\theta_1 A_1 T_w}{(A_1^2+2B_2^2)} + \frac{\theta_1 A_2 T_w}{(A_2^2+2B_2^2)} \Big] - \frac{2RCB_2^2}{P} \Big[\theta_1^2 \Big[\frac{A_1^2(2A_1^2+6B_2^2)}{16B_2^4(2B_2^2+A_1^2)^{3/2}} - \frac{A_1^3}{3} \frac{A_1(60B_2^4 + 40A_1B_2^2 + 8A_1^4)}{64B_2^6(2B_2^2+A_1^2)^{5/2}}$$

$$+ \frac{A_2^5}{10} \frac{A_2}{16(B_1^2+B_2^2)^4(B_1^2+B_2^2+A_2^2)^{7/2}} \Big[105(B_1^2+B_2^2)^3 + 210(B_1^2+B_2^2)^2 A_2^2 + 168(B_1^2+B_2^2)A_2^4 + 48A_2^6 \Big] - \ldots$$

$$+ 2520(B_1^2+B_2^2)^3 A_2^2 + 3024(B_1^2+B_2^2)^2 A_2^4 + 1728(B_1^2+B_2^2)A_2^6 + 384A_2^8 \Big] \Big] + 2\theta_1^2 \Big[\frac{A_1 A_2(2A \ldots}{4(2B_2^2)^2(2 \ldots}$$

$$+ \frac{A_1^5}{10} \frac{A_2}{16(2B_2^2)^4(2B_2^2+A_2^2)^{7/2}} \Big[105(2B_2^2)^3 + 210(2B_2^2)^2 A_2^2 + 168(2B_2^2)A_2^4 + 48A_2^6 \Big] - \frac{A_1^7}{42} \frac{A_2}{32(2B_2^2)^5(2 \ldots}$$

$$+ 3024(2B_2^2)^2 A_2^4 + 1728(2B_2^2)A_2^6 + 384A_2^8 \Big] \Big] + \theta_1^2 \Big[\frac{A_2^2(2A_2^2+6B_2^2)}{16B_2^4(2B_2^2+A_2^2)^{3/2}} - \frac{A_2^3}{3} \frac{A_2(60B_2^4 + \ldots}{64B_2^6(2B_2^2 + \ldots}$$

$$+ \frac{A_2^5}{10} \frac{A_2}{16(2B_2^2)^4(2B_2^2+A_2^2)^{7/2}} \Big[105(2B_2^2)^3 + 210(2B_2^2)^2 A_2^2 + 168(2B_2^2)A_2^4 + 48A_2^6 \Big] - \frac{A_2^7}{42} \frac{A \ldots}{32(2B_2^2)^5(A \ldots}$$

$$+ 3024(2B_2^2)^2 A_2^4 + 1728(2B_2^2)A_2^6 + 384A_2^8 \Big] \Big] + \frac{\theta_1 T_w}{\sqrt{\pi}} \Big[\frac{1}{(2B_2^2)^{3/2}} \tan^{-1}\Big(\frac{A_1}{\sqrt{2}\,B_2} \Big) + \frac{A_1}{(2B_2^2)(B_1^2 + \ldots}$$

$$+ \frac{A_2}{(2B_2^2)(B_1^2+2B_2^2)} \Big] + \frac{\sqrt{2\pi}}{16} \frac{T_w^2}{B_2^3} \Big] + \frac{C\eta A_1 A_2}{\sqrt{\pi}(A_1^2+2B_2^2)} + \frac{C\eta\theta_1 A_2}{\sqrt{\pi}(A_2^2+2B_2^2)} - 2C\eta B_2^2 \Big[\frac{\theta_1}{2\sqrt{\pi}} \Big[\frac{1}{(2B_2^2)^{3/2} \ldots}$$

$$+ 1728(B_1^2+B_2^2)A_2^6 + 384 A_2^4 \Big] \Big] + \theta_1^2 \left[\frac{A_2^2(2A_2^2 + 3(B_1^2+B_2^2))}{4(B_1^2+B_2^2)^2(B_1^2+B_2^2+A_2^2)^{3/2}} \right.$$

$$+ 8A_2^4 \Big] + \frac{A_2^5}{10} \frac{A_2}{16(B_1^2+B_2^2)^4(B_1^2+B_2^2+A_2^2)^{7/2}} \Big[105(B_1^2+B_2^2)^3 + 210(B_1^2+B_2^2)^2 A_2^2 + 168(B_1^2+B_2^2)A_2^4$$

$$)^{9/2} \Big[945(B_1^2+B_2^2)^4 + 2520(B_1^2+B_2^2)^3 A_2^2 + 3024(B_1^2+B_2^2)^2 A_2^4 + 1728(B_1^2+B_2^2)A_2^6 + 384 A_2^8 \Big]\Big]$$

$$\frac{A_1}{(B_1^2+B_2^2)(A_1^2+B_1^2+B_2^2)} \Big] + \frac{\theta_1 T_w}{\sqrt{\pi}} \left[\frac{1}{(B_1^2+B_2^2)^{3/2}} \tan^{-1}\left(\frac{A_2}{(B_1^2+B_2^2)^{1/2}}\right) + \frac{A_2}{(B_1^2+B_2^2)(A_2^2+B_1^2+B_2^2)} \right]$$

$$\theta_1 A_1 \frac{1}{(A_1^2+B_1^2+B_2^2)} + \frac{1}{\sqrt{\pi}} C\eta\,\theta_1 A_2 \frac{1}{(A_2^2+B_1^2+B_2^2)} - 2\eta C B_1^2 \left[\frac{\theta_1}{2\sqrt{\pi}} \left[\frac{1}{(B_1^2+B_2^2)^{3/2}} \tan^{-1}\left(\frac{A_1}{(B_1^2+B_2^2)^{1/2}}\right) \right.\right.$$

$$B_2^2)^{3/2} \tan^{-1}\left(\frac{A_2}{(B_1^2+B_2^2)^{1/2}}\right) + \frac{A_2}{(B_1^2+B_2^2)(A_2^2+B_1^2+B_2^2)} \Big] + T_w \frac{\sqrt{\pi}}{4(B_1^2+B_2^2)^{1/2}(B_1^2+B_2^2)} \Big] \Big\}$$

$$B_2^2)^{1/2} + \frac{\theta_1^2 A_1 A_2}{\sqrt{\pi}(A_2^2+2B_2^2)(A_1^2+A_2^2+2B_2^2)^{1/2}} + \frac{\theta_1^2 A_1 A_2}{(A_1^2+2B_2^2)(A_1^2+A_2^2+2B_2^2)^{1/2}} + \frac{\sqrt{2}\,\theta_1^2 A_2^2}{(A_2^2+2B_2^2)(A_2^2+B_2^2)^{1/2}}$$

$$\left[\frac{A_1^2(2A_1^2+6B_2^2)}{16B_2^4(2B_2^2+A_1^2)^{3/2}} - \frac{A_1^3}{3} \frac{A_1(60B_2^4 + 40A_1B_2^2 + 8A_1^4)}{64B_2^6(2B_2^2+A_1^2)^{5/2}} \right.$$

$$+ B_2^2)^3 + 210(B_1^2+B_2^2)^2 A_2^2 + 168(B_1^2+B_2^2)A_2^4 + 48A_2^6 \Big] - \frac{A_2^7}{42} \frac{A_2}{32(B_1^2+B_2^2)^5(B_1^2+B_2^2+A_2^2)^{9/2}} \Big[945(B_1^2+B_2^2)^4$$

$$+ 1728(B_1^2+B_2^2)A_2^6 + 384A_2^8 \Big] \Big] + 2\theta_1^2 \left[\frac{A_1 A_2(2A_2^2+6B_2^2)}{4(2B_2^2)^2(2B_2^2+A_2^2)^{3/2}} - \frac{A_1^3}{3} \frac{A_2(15(2B_2^2)^2 + 20(2B_2)A_2 + 8A_2^4)}{8(2B_2^2)^3(2B_2^2+A_2^2)^{5/2}} \right.$$

$$210(2B_2^2)^2 A_2^2 + 168(2B_2^2)A_2^4 + 48A_2^6 \Big] - \frac{A_1^7}{42} \frac{A_2}{32(2B_2^2)^5(2B_2^2+A_2^2)^{9/2}} \Big[945(2B_2^2)^4 + 2520(2B_2^2)^3 A_2^2$$

$$84 A_2^8 \Big] \Big] + \theta_1^2 \left[\frac{A_2^2(2A_2^2+6B_2^2)}{16B_2^4(2B_2^2+A_2^2)^{3/2}} - \frac{A_2^3}{3} \frac{A_2(60B_2^4 + 40A_2B_2^2 + 8A_2^4)}{64B_2^6(2B_2^2+A_2^2)^{5/2}} \right.$$

$$210(2B_2^2)^2 A_2^2 + 168(2B_2^2)A_2^4 + 48A_2^6 \Big] - \frac{A_2^7}{42} \frac{A_2}{32(2B_2^2)^5(A_2^2+2B_2^2)^{9/2}} \Big[945(2B_2^2)^4 + 2520(2B_2^2)^3 A_2^2$$

$$4 A_2^8 \Big]\Big] + \frac{\theta_1 T_w}{\sqrt{\pi}} \left[\frac{1}{(2B_2^2)^{3/2}} \tan^{-1}\left(\frac{A_1}{\sqrt{2}\,B_2}\right) + \frac{A_1}{(2B_2^2)(B_1^2+2B_2^2)} \right] + \frac{\theta_1 T_w}{\sqrt{\pi}} \left[\frac{1}{(2B_2^2)^{3/2}} \tan^{-1}\left(\frac{A_2}{\sqrt{2}\,B_2}\right) \right.$$

$$\frac{A_1 A_2}{+2B_2^2)} + \frac{C\eta\,\theta_1 A_2}{\sqrt{\pi}(A_2^2+2B_2^2)} - 2C\eta B_2^2 \left[\frac{\theta_1}{2\sqrt{\pi}} \left[\frac{1}{(2B_2^2)^{3/2}} \tan^{-1}\left(\frac{A_1}{\sqrt{2}\,B_2}\right) + \frac{A_1}{(2B_2^2)(B_1^2+2B_2^2)} \right] \right.$$

$$+\frac{\theta_1}{2\sqrt{\pi}}\left[\frac{1}{(2B_2^2)^{1/2}}\tan^{-1}\left(\frac{A_2}{\sqrt{2}\,B_2}\right)+\frac{A_2}{(2B_2^2)(B_1^2+2B_2^2)}\right]+\frac{\sqrt{2\pi}}{16}\frac{T_w}{B_2^2}\Big]\Big\}\Big\}\Big/0.25\sqrt{\frac{2}{\pi}}\,\Omega_2^2\frac{\mu_1^2}{B_2^3} \qquad [3.24]$$

The following additional information is necessary to perform the analysis similar to the above on Equation 3.6.

$$\frac{\partial\theta}{\partial t}=\delta_1\frac{\partial\theta_1}{\partial t}\,\mathrm{erf}(A_1\bar{y})+\frac{2}{\sqrt{\pi}}\delta_1\theta_1A_1\frac{\partial\bar{y}}{\partial t}e^{-A_1^2\bar{y}^2}+\frac{2}{\sqrt{\pi}}\delta_1\theta_1\bar{y}\frac{\partial A_1}{\partial t}e^{-A_1^2\bar{y}^2}+\delta_2\frac{\partial\theta_1}{\partial t}\,\mathrm{erf}(A_2\bar{y})+\frac{2}{\sqrt{\pi}}\delta_2\theta_1A_2\frac{\partial\bar{y}}{\partial t}e^{-A_2^2\bar{y}^2}+\frac{2}{\sqrt{\pi}}\delta_2\theta_1\bar{y}\frac{\partial A_2}{\partial t}e^{-A_2^2\bar{y}^2} \qquad [3.25]$$

$$\frac{\partial\theta}{\partial x}=\delta_1\frac{\partial\theta_1}{\partial x}\,\mathrm{erf}(A_1\bar{y})+\frac{2}{\sqrt{\pi}}\delta_1\theta_1\bar{y}\frac{\partial A_1}{\partial x}e^{-A_1^2\bar{y}^2}+\delta_2\frac{\partial\theta_1}{\partial x}\,\mathrm{erf}(A_2\bar{y})+\frac{2}{\sqrt{\pi}}\delta_2\theta_1\bar{y}\frac{\partial A_2}{\partial x}e^{-A_2^2\bar{y}^2} \qquad [3.26]$$

$$\frac{\partial\theta}{\partial\bar{y}}=\frac{2}{\sqrt{\pi}}\delta_1\theta_1A_1e^{-A_1^2\bar{y}^2}+\frac{2}{\sqrt{\pi}}\delta_2\theta_1A_2e^{-A_2^2\bar{y}^2} \qquad [3.27]$$

Formation of the residual equation for the Energy Equation 3.6 is analogous to the formation of the Momentum Equation residual. Analogously, 2 Energy Equations are then formed by multiplying the residual Energy Equation by its two respective weighting functions, $(2/\sqrt{\pi})\delta_1\bar{y}\,e^{-A_1^2\bar{y}^2}$ and $(2/\sqrt{\pi})\delta_2\bar{y}\,e^{-A_2^2\bar{y}^2}$. As this expansion becomes very lengthy, details of the entire expansion are bypassed. After one obtains the residual equations and performs the integrations over $\bar{y}$, exactly as in the Momentum Equations, these 2 Energy Equations become

$$\begin{aligned}\frac{\partial A_1}{\partial t}=\Big\{&-\frac{1}{2}\sqrt{\frac{2}{\pi}}\delta_1^2C_p\theta_1\frac{\partial\theta_1}{\partial t}\frac{1}{A_1^2}-\frac{1}{\sqrt{\pi}}\delta_1\delta_2C_p\theta_1\frac{\partial\theta_1}{\partial t}\frac{A_2}{(A_1^2+A_2^2)^{1/2}}-\frac{1}{\sqrt{\pi}}C_p\theta_1^2\frac{\partial A_2}{\partial t}\frac{\delta_1\delta_2}{(A_1^2+A_2^2)^{1/2}(A_1^2+A_2^2)}-\frac{1}{\sqrt{\pi}}\delta_1C_p\theta_1\frac{\partial T_w}{\partial t}\frac{1}{A_1^2}\\&-\frac{2\mu_1}{\pi\sqrt{\pi}}\delta_1^2C_p\theta_1\frac{\partial\theta_1}{\partial x}\left[\frac{1}{\sqrt{2}A_1}\tan^{-1}\left(\frac{B_1}{\sqrt{2}A_1}\right)+\frac{B_1}{A_1^2(A_1^2+B_1^2)^{1/2}}\tan^{-1}\left(\frac{A_1}{(A_1^2+B_1^2)^{1/2}}\right)\right]-\frac{2}{\pi\sqrt{\pi}}\Omega_1\delta_1\delta_2C_p\theta_1\frac{\partial\theta_1}{\partial x}\mu_1\Big[\frac{A_2}{A_1(A_1^2+A_2^2)^{1/2}}\tan^{-1}\left(\frac{B_1}{(A_1^2+A_2^2)^{1/2}}\right)\\&+\frac{B_1}{A_1^2(A_1^2+B_1^2)^{1/2}}\tan^{-1}\left(\frac{A_2}{(A_1^2+B_1^2)^{1/2}}\right)\Big]-\frac{\Omega_1\delta_1^2}{\pi\sqrt{\pi}}C_p\theta_1^2\mu_1\frac{\partial A_1}{\partial x}\left[\frac{1}{\sqrt{2}A_1^3}\tan^{-1}\left(\frac{B_1}{\sqrt{2}A_1}\right)+\frac{B_1}{A_1^2(B_1^2+2A_1^2)}\right]\\&-\frac{2}{\pi\sqrt{\pi}}\Omega_1\delta_1\delta_2C_p\theta_1^2\mu_1\frac{\partial A_2}{\partial x}\left[\frac{1}{(A_1^2+A_2^2)^{3/2}}\tan^{-1}\left(\frac{B_1}{(A_1^2+A_2^2)^{1/2}}\right)+\frac{B_1}{(A_1^2+A_2^2)(A_1^2+A_2^2+B_1^2)}\right]-\frac{2}{\pi\sqrt{\pi}}\Omega_2\delta_1^2C_p\theta_1\frac{\partial\theta_1}{\partial x}\mu_1\Big[\frac{1}{\sqrt{2}A_1^2}\tan^{-1}\left(\frac{B_2}{\sqrt{2}A_1}\right)\\&+\frac{B_2}{A_1^2(A_1^2+B_2^2)^{1/2}}\tan^{-1}\left(\frac{A_1}{(A_1^2+B_2^2)^{1/2}}\right)\Big]-\frac{2}{\pi\sqrt{\pi}}\Omega_1\delta_1\delta_2C_p\theta_1\frac{\partial\theta_1}{\partial x}\mu_1\left[\frac{A_2}{A_1^2(A_1^2+A_2^2)^{1/2}}\tan^{-1}\left(\frac{B_2}{(A_1^2+A_2^2)^{1/2}}\right)+\frac{B_2}{A_1^2(A_1^2+B_2^2)^{1/2}}\tan^{-1}\left(\frac{A_2}{(A_1^2+B_2^2)^{1/2}}\right)\right]\Big]\\&-\frac{2}{\pi\sqrt{\pi}}\Omega_2\delta_1^2C_p\theta_1^2\frac{\partial A_1}{\partial x}\mu_1\left[\frac{1}{\sqrt{2}A_1^3}\tan^{-1}\left(\frac{B_2}{\sqrt{2}A_1}\right)+\frac{B_2}{A_1^2(B_2^2+2A_1^2)}\right]-\frac{2}{\pi\sqrt{\pi}}\Omega_2\delta_1\delta_2C_p\theta_1^2\mu_1\frac{\partial A_2}{\partial x}\Big[\frac{1}{(A_1^2+A_2^2)^{3/2}}\tan^{-1}\left(\frac{B_2}{(A_1^2+A_2^2)^{1/2}}\right)\end{aligned}$$

$$+ \frac{\theta_1}{2\sqrt{\pi}} \left[\frac{1}{(2B_2^2)^{3/2}} \tan^{-1}\left(\frac{A_2}{\sqrt{2}B_2}\right) + \frac{A_2}{(2B_2^2)(B_1^2 + 2B_2^2)} \right] + \frac{\sqrt{2\pi}}{16} \frac{T_w}{B_2^3} \Big] \Big\} \Big\} \Big/ 0.25\sqrt{\frac{2}{\pi}}$$

The following additional information is necessary to perform the analysis similar

$$\frac{\partial\theta}{\partial t} = \delta_1 \frac{\partial\theta_1}{\partial t} \operatorname{erf}(A_1\bar{y}) + \frac{2}{\sqrt{\pi}} \delta_1 \theta_1 A_1 \frac{\partial\bar{y}}{\partial t} e^{-A_1^2\bar{y}^2} + \frac{2}{\sqrt{\pi}} \delta_1 \theta_1 \bar{y} \frac{\partial A_1}{\partial t} e^{-A_1^2\bar{y}^2} + \delta_2 \frac{\partial\theta_1}{\partial t} \operatorname{erf}(A_2\bar{y}) + \frac{2}{\sqrt{\pi}} \delta_2$$

$$\frac{\partial\theta}{\partial x} = \delta_1 \frac{\partial\theta_1}{\partial x} \operatorname{erf}(A_1\bar{y}) + \frac{2}{\sqrt{\pi}} \delta_1 \theta_1 \bar{y} \frac{\partial A_1}{\partial x} e^{-A_1^2\bar{y}^2} + \delta_2 \frac{\partial\theta_1}{\partial x} \operatorname{erf}(A_2\bar{y}) + \frac{2}{\sqrt{\pi}} \delta_2 \theta_1 \bar{y} \frac{\partial A_2}{\partial x} e^{-A_2^2}$$

$$\frac{\partial\theta}{\partial\bar{y}} = \frac{2}{\sqrt{\pi}} \delta_1 \theta_1 A_1 e^{-A_1^2\bar{y}^2} + \frac{2}{\sqrt{\pi}} \delta_2 \theta_1 A_2 e^{-A_2^2\bar{y}^2}$$

Formation of the residual equation for the Energy Equation 3.6 is analogous to the residual. Analogously, 2 Energy Equations are then formed by multiplying the res respective weighting functions, $(2/\sqrt{\pi})\,\delta_1\,\bar{y}\,e^{-A_1^2\bar{y}^2}$ and $(2/\sqrt{\pi})\,\delta_2\,\bar{y}\,e^{-A_2^2\bar{y}^2}$. As details of the entire expansion are bypassed. After one obtains the residual equ over $\bar{y}$, exactly as in the Momentum Equations, these 2 Energy Equations become

$$\frac{\partial A_1}{\partial t} = \Big\{ -\frac{1}{2}\sqrt{\frac{2}{\pi}}\, \delta_1^2 C_p \theta_1 \frac{\partial\theta_1}{\partial t} \frac{1}{A_1^2} - \frac{1}{\sqrt{\pi}} \delta_1\delta_2 C_p \theta_1 \frac{\partial\theta_1}{\partial t} \frac{A_2}{(A_1^2+A_2^2)^{1/2}} - \frac{1}{\sqrt{\pi}} C_p \theta_1^2 \frac{\partial A_2}{\partial t} \frac{\delta_1\delta_2}{(A_1^2+A_2^2)^{1/2}(A_1^2}$$

$$- \frac{2\mu_1}{\pi\sqrt{\pi}} \delta_1^2 C_p \theta_1 \frac{\partial\theta_1}{\partial x} \left[\frac{1}{\sqrt{2}A_1} \tan^{-1}\left(\frac{B_1}{\sqrt{2}A_1}\right) + \frac{B_1}{A_1^2(A_1^2+B_1^2)^{1/2}} \tan^{-1}\left(\frac{A_1}{(A_1^2+B_1^2)^{1/2}}\right) \right] - \frac{2}{\pi\sqrt{\pi}} \Omega_1$$

$$+ \frac{B_1}{A_1^2(A_1^2+B_1^2)^{1/2}} \tan^{-1}\left(\frac{A_2}{(A_1^2+B_1^2)^{1/2}}\right)\Big] - \frac{\Omega_1\delta_1^2}{\pi\sqrt{\pi}} C_p \theta_1^2 \mu_1 \frac{\partial A_1}{\partial x} \left[\frac{1}{\sqrt{2}A_1^3} \tan^{-1}\left(\frac{B_1}{\sqrt{2}A_1}\right) + \frac{}{A_1^2(B}\right.$$

$$- \frac{2}{\pi\sqrt{\pi}} \Omega_1 \delta_1\delta_2 C_p \theta_1^2 \mu_1 \frac{\partial A_2}{\partial x} \left[\frac{1}{(A_1^2+A_2^2)^{3/2}} \tan^{-1}\left(\frac{B_1}{(A_1^2+A_2^2)^{1/2}}\right) + \frac{B_1}{(A_1^2+A_2^2)(A_1^2+A_2^2+B_1^2)}\right.$$

$$+ \frac{B_2}{A_1^2(A_1^2+B_2^2)^{1/2}} \tan^{-1}\left(\frac{A_1}{(A_1^2+B_2^2)^{1/2}}\right)\Big] - \frac{2}{\pi\sqrt{\pi}} \Omega_1 \delta_1\delta_2 C_p \theta_1 \frac{\partial\theta_1}{\partial x} \mu_1 \left[\frac{A_2}{A_1^2(A_1^2+A_2^2)^{1/2}} \tan\right.$$

$$- \frac{2}{\pi\sqrt{\pi}} \Omega_2 \delta_1^2 C_p \theta_1^2 \frac{\partial A_1}{\partial x} \mu_1 \left[\frac{1}{\sqrt{2}A_1^3} \tan^{-1}\left(\frac{B_2}{\sqrt{2}A_1}\right) + \frac{B_2}{A_1^2(B_2^2+2A_1^2)} \right] - \frac{2}{\pi\sqrt{\pi}} \Omega_2 \delta_1\delta_2 C_p \theta_1^2 \mu$$

$$\frac{A_2}{B_1^2 + 2B_2^2)}\Big] + \frac{\sqrt{2\pi}}{16}\frac{T_w}{B_2^3}\Big]\Big\}\Big\}\Big/ 0.25\sqrt{\frac{2}{\pi}}\,\Omega_2^2\frac{\mu_1^2}{B_2^3} \qquad [3.24]$$

essary to perform the analysis similar to the above on Equation 3.6.

$$\delta_1\theta_1\bar\eta\frac{\partial A_1}{\partial t}e^{-A_1^2\bar\eta^2} + \delta_2\frac{\partial\theta_1}{\partial t}\,\mathrm{erf}(A_2\bar\eta) + \frac{2}{\sqrt\pi}\delta_2\theta_1A_2\frac{\partial\bar\eta}{\partial t}e^{-A_2^2\bar\eta^2} + \frac{2}{\sqrt\pi}\delta_2\theta_1\bar\eta\frac{\partial A_2}{\partial t}e^{-A_2^2\bar\eta^2} \qquad [3.25]$$

$$+\ \delta_2\frac{\partial\theta_1}{\partial x}\,\mathrm{erf}(A_2\bar\eta) + \frac{2}{\sqrt\pi}\delta_2\theta_1\bar\eta\frac{\partial A_2}{\partial x}e^{-A_2^2\bar\eta^2} \qquad [3.26]$$

$$[3.27]$$

Energy Equation 3.6 is analogous to the formation of the Momentum Equation
are then formed by multiplying the residual Energy Equation by its two
$A_1^2\bar\eta^2$ and $(2/\sqrt\pi)\,\delta_2\bar\eta\,e^{-A_2^2\bar\eta^2}$. As this expansion becomes very lengthy,
d. After one obtains the residual equations and performs the integrations
s, these 2 Energy Equations become

$$\frac{\partial\theta_1}{\partial t}\frac{A_2}{(A_1^2+A_2^2)^{1/2}} - \frac{1}{\sqrt\pi}C_p\theta_1^2\frac{\partial A_2}{\partial t}\frac{\delta_1\delta_2}{(A_1^2+A_2^2)^{1/2}(A_1^2+A_2^2)} - \frac{1}{\sqrt\pi}\delta_1C_p\theta_1\frac{\partial T_w}{\partial t}\frac{1}{A_1^2}$$

$$\frac{B_1}{A_1^2(A_1^2+B_1^2)^{1/2}}\tan^{-1}\left(\frac{A_1}{(A_1^2+B_1^2)^{1/2}}\right)\Big] - \frac{2}{\pi\sqrt\pi}\Omega_1\delta_1\delta_2C_p\theta_1\frac{\partial\theta_1}{\partial x}\mu_1\Big[\frac{A_2}{A_1(A_1^2+A_2^2)^{1/2}}\tan^{-1}\left(\frac{B_1}{(A_1^2+A_2^2)^{1/2}}\right)$$

$$\frac{\delta_1^2}{\sqrt\pi}C_p\theta_1^2\mu_1\frac{\partial A_1}{\partial x}\Big[\frac{1}{\sqrt2 A_1^3}\tan^{-1}\left(\frac{B_1}{\sqrt2A_1}\right) + \frac{B_1}{A_1^2(B_1^2+2A_1^2)}\Big]$$

$$\tan^{-1}\left(\frac{B_1}{(A_1^2+A_2^2)^{1/2}}\right) + \frac{B_1}{(A_1^2+A_2^2)(A_1^2+A_2^2+B_1^2)}\Big] - \frac{2}{\pi\sqrt\pi}\Omega_2\delta_1^2C_p\theta_1\frac{\partial\theta_1}{\partial x}\mu_1\Big[\frac{1}{\sqrt2A_1^2}\tan^{-1}\left(\frac{B_2}{\sqrt2A_1}\right)$$

$$\frac{2}{\pi\sqrt\pi}\Omega_1\delta_1\delta_2C_p\theta_1\frac{\partial\theta_1}{\partial x}\mu_1\Big[\frac{A_2}{A_1^2(A_1^2+A_2^2)^{1/2}}\tan^{-1}\left(\frac{B_2}{(A_1^2+A_2^2)^{1/2}}\right) + \frac{B_2}{A_1^2(A_1^2+B_2^2)^{1/2}}\tan^{-1}\left(\frac{A_2}{(A_1^2+B_2^2)^{1/2}}\right)\Big]$$

$$\frac{B_2}{2A_1}\Big) + \frac{B_2}{A_1^2(B_2^2+2A_1^2)}\Big] - \frac{2}{\pi\sqrt\pi}\Omega_2\delta_1\delta_2C_p\theta_1^2\mu_1\frac{\partial A_2}{\partial x}\Big[\frac{1}{(A_1^2+A_2^2)^{3/2}}\tan^{-1}\left(\frac{B_2}{(A_1^2+A_2^2)^{1/2}}\right)$$

$$+ \frac{B_2}{(A_1^2+A_2^2)(A_1^2+A_2^2+B_2^2)} \Big] - \frac{\Omega_1 \delta_1}{\sqrt{\pi}} C_p \theta_1 \mu_1 \frac{\partial T_w}{\partial x} \frac{B_1}{A_1^2(A_1^2+B_1^2)^{1/2}} - \frac{\Omega_2 \delta_1}{\sqrt{\pi}} C_p \theta_1 \mu_1 \frac{\partial T_w}{\partial x} \frac{B_2}{A_1^2(A_1^2+B_2^2)^{1/2}} + \frac{2}{\pi\sqrt{\pi}} \Omega_1 \delta_1^2 C_p \theta_1^2 \frac{\partial \mu_1}{\partial x} A_1 \Big[\frac{1}{\sqrt{2}\,A_1^2} \tan^{-1}\left(\frac{B_1}{\sqrt{2}\,A_1}\right)$$

$$+ \frac{B_1}{A_1^2(B_1^2+2A_1^2)} \Big] + \frac{2}{\pi\sqrt{\pi}} \Omega_1 \delta_1 \delta_2 C_p \theta_1^2 \frac{\partial \mu_1}{\partial x} A_2 \Big[\frac{1}{(A_1^2+A_2^2)^{3/2}} \tan^{-1}\left(\frac{B_1}{(A_1^2+A_2^2)^{1/2}}\right) + \frac{B_1}{(A_1^2+A_2^2)(B_1^2+A_1^2+A_2^2)} \Big]$$

$$+ \frac{2}{\pi\sqrt{\pi}} \Omega_2 \delta_1^2 C_p \theta_1^2 \frac{\partial \mu_1}{\partial x} A_1 \Big[\frac{1}{\sqrt{2}\,A_1^2} \tan^{-1}\left(\frac{B_2}{\sqrt{2}\,A_1}\right) + \frac{B_2}{A_1^2(B_2^2+2A_1^2)} \Big] + \frac{2}{\pi\sqrt{\pi}} \Omega_2 \delta_1 \delta_2 C_p \theta_1^2 \frac{\partial \mu_1}{\partial x} A_2 \Big[\frac{1}{(A_1^2+A_2^2)^{3/2}} \tan^{-1}\left(\frac{B_2}{(A_1^2+A_2^2)^{1/2}}\right)$$

$$+ \frac{B_2}{(A_2^2+A_1^2)(B_2^2+A_1^2+A_2^2)} \Big] + \frac{4}{\pi\sqrt{\pi}} C_p \theta_1^2 \mu_1 \frac{\partial B_1}{\partial x} A_1 \frac{\Omega_1 \delta_1^2}{(2A_1^2+B_1^2)^2} + \frac{4}{\pi\sqrt{\pi}} C_p \theta_1^2 \mu_1 \frac{\partial B_1}{\partial x} A_2 \frac{\Omega_1 \delta_1 \delta_2}{(A_1^2+A_2^2+B_1^2)^2} + \frac{2}{\pi\sqrt{\pi}} C_p \theta_1^2 \frac{\partial \mu_1}{\partial x} \frac{A_1}{B_1} \frac{\Omega_1 \delta_1^2}{(2A_1^2+B_1^2)}$$

$$+ \frac{2}{\pi} C_p \theta_1^2 \frac{\partial \mu_1}{\partial x} \frac{A_2}{B_1} \frac{\Omega_1 \delta_1 \delta_2}{(A_1^2+A_2^2+B_1^2)^{1/2}} - \frac{2}{\pi\sqrt{\pi}} C_p \theta_1^2 \mu_1 \frac{\partial B_1}{\partial x} \frac{A_1}{B_1^2} \frac{\Omega_1 \delta_1^2}{(2A_1^2+B_1^2)} - \frac{2}{\pi\sqrt{\pi}} C_p \theta_1^2 \mu_1 \frac{\partial B_1}{\partial x} \frac{A_2}{B_1^2} \frac{\Omega_1 \delta_1 \delta_2}{(A_1^2+A_2^2+B_1^2)}$$

$$- \frac{4}{\pi\sqrt{\pi}} C_p \theta_1^2 \mu_1 A_1 \frac{\partial B_1}{\partial x} \frac{\Omega_1 \delta_1^2}{(2A_1^2+B_1^2)^2} - \frac{4}{\pi\sqrt{\pi}} C_p \theta_1^2 \mu_1 A_2 \frac{\partial B_1}{\partial x} \frac{\Omega_1 \delta_1 \delta_2}{(A_1^2+A_2^2+B_1^2)^2} + \frac{4}{\pi\sqrt{\pi}} C_p \theta_1^2 \mu_1 A_1 \frac{\partial B_2}{\partial x} \frac{\Omega_2 \delta_1^2}{(2A_1^2+B_2^2)^2}$$

$$+ \frac{4}{\pi\sqrt{\pi}} C_p \theta_1^2 \mu_1 A_2 \frac{\partial B_2}{\partial x} \frac{\Omega_2 \delta_1 \delta_2}{(A_1^2+A_2^2+B_2^2)^2} + \frac{2}{\pi\sqrt{\pi}} C_p \theta_1^2 \frac{\partial \mu_1}{\partial x} \frac{A_1}{B_2} \frac{\Omega_2 \delta_1^2}{(2A_1^2+B_2^2)} + \frac{2}{\pi\sqrt{\pi}} C_p \theta_1^2 \frac{\partial \mu_1}{\partial x} \frac{A_2}{B_2} \frac{\Omega_2 \delta_1 \delta_2}{(A_1^2+A_2^2+B_2^2)}$$

$$- \frac{2}{\pi\sqrt{\pi}} C_p \theta_1^2 \mu_1 \frac{\partial B_2}{\partial x} \frac{A_1}{B_2^2} \frac{\Omega_2 \delta_1^2}{(2A_1^2+B_2^2)} - \frac{2}{\pi\sqrt{\pi}} C_p \theta_1^2 \mu_1 \frac{\partial B_2}{\partial x} \frac{A_2}{B_2^2} \frac{\Omega_2 \delta_1 \delta_2}{(A_1^2+A_2^2+B_2^2)} - \frac{4}{\pi\sqrt{\pi}} C_p \theta_1^2 \mu_1 A_1 \frac{\partial B_2}{\partial x} \frac{\Omega_1 \delta_1^2}{(2A_1^2+B_2^2)^2}$$

$$- \frac{4}{\pi\sqrt{\pi}} C_p \theta_1^2 \mu_1 A_2 \frac{\partial B_2}{\partial x} \frac{\Omega_2 \delta_1 \delta_2}{(A_1^2+A_2^2+B_2^2)^2} - \frac{1}{\pi\sqrt{\pi}} C_p \theta_1^2 \frac{\partial \mu_1}{\partial x} \frac{\Omega_1 \delta_1^2}{A_1 B_1} - \frac{2}{\pi\sqrt{\pi}} C_p \theta_1^2 \frac{\partial \mu_1}{\partial x} \frac{A_2}{B_1} \frac{\Omega_1 \delta_1 \delta_2}{(A_1^2+A_2^2)} + \frac{1}{\pi\sqrt{\pi}} C_p \theta_1^2 \mu_1 \frac{\partial B_1}{\partial x} \frac{\Omega_1 \delta_1^2}{A_1 B_1^2}$$

$$+ \frac{2}{\pi\sqrt{\pi}} C_p \theta_1^2 \mu_1 \frac{\partial B_1}{\partial x} \frac{A_2}{B_1^2} \frac{\Omega_1 \delta_1 \delta_2}{(A_1^2+A_2^2)} - \frac{\Omega_2 \delta_1^2}{\pi\sqrt{\pi}} C_p \theta_1^2 \frac{\partial \mu_1}{\partial x} \frac{1}{A_1 B_2} - \frac{2}{\pi\sqrt{\pi}} C_p \theta_1^2 \frac{\partial \mu_1}{\partial x} \frac{A_2}{B_2} \frac{\Omega_2 \delta_1 \delta_2}{(A_1^2+A_2^2)} + \frac{1}{\pi\sqrt{\pi}} C_p \theta_1^2 \mu_1 \frac{\partial B_2}{\partial x} \frac{\Omega_2 \delta_1^2}{A_1 B_2^2}$$

$$+ \frac{2}{\pi\sqrt{\pi}} C_p \theta_1^2 \mu_1 \frac{\partial B_2}{\partial x} \frac{A_2}{B_2^2} \frac{\Omega_2 \delta_1 \delta_2}{(A_1^2+A_2^2)} - \frac{1}{2}\sqrt{\frac{2}{\pi}}\, C_p \frac{C T_1}{P R} \theta_1^2 \frac{\rho_1}{\rho_0^2} \delta_1^2 - \frac{2}{\sqrt{\pi}} \frac{C_p \theta_1^2 A_2^2 \rho C T_1 \delta_1 \delta_2}{\rho_0^2 P_r (A_1^2+A_2^2)(A_1^2+A_2^2)^{1/2}} + \frac{4}{\pi\sqrt{\pi}} \frac{\Omega_1^2 \delta_1 \theta_1 \mu_1^2 B_1^2 C T_1}{(A_1^2+2B_1^2)} \frac{\rho_1}{\rho_0^2}$$

$$+ \frac{8}{\pi\sqrt{\pi}} \frac{\Omega_1 \Omega_2 \delta_1 \theta_1 \mu_1^2 B_1 B_2 C T_1}{(A_1^2+B_1^2+B_2^2)} \frac{\rho_1}{\rho_0^2} + \frac{4}{\pi\sqrt{\pi}} \frac{\Omega_2^2 \delta_1 \theta_1 \mu_1^2 B_2^2 C T_1}{(A_1^2+2B_2^2)} \frac{\rho_1}{\rho_0^2} + \frac{1}{2}\sqrt{\frac{2}{\pi}} \frac{\theta_1^2 C_p R}{R\theta_1+\eta P+RT_w} \frac{\partial T_1}{\partial t} \frac{\delta_1^2}{A_1^2}$$

$$+ \frac{1}{\sqrt{\pi}} C_p \theta_1^2 \frac{\partial T_1}{\partial t} \frac{A_2}{A_1^2(A_1^2+A_2^2)^{1/2}} \frac{R \delta_1 \delta_2}{R\theta_1+\eta P+RT_w} + \frac{1}{\sqrt{\pi}} C_p \theta_1 \frac{\partial T_1}{\partial t} \frac{\eta P+RT_w}{R\theta_1+\eta P+RT_w} \frac{\delta_1}{A_1^2} + \frac{1}{2}\sqrt{\frac{2}{\pi}}\, C_p \theta_1^2 \mu_1 \frac{\partial T_1}{\partial x} \frac{R}{R\theta_1+\eta P+RT_w} \frac{\delta_1^2}{A_1^2}$$

$$+ \frac{1}{\sqrt{\pi}} C_p \theta_1^2 \mu_1 \frac{\partial T_1}{\partial x} \frac{R \delta_1 \delta_2}{R\theta_1+\eta P+RT_w} \frac{A_2}{A_1^2(A_1^2+A_2^2)^{1/2}} + \frac{1}{\sqrt{\pi}} C_p \theta_1 \mu_1 \frac{\partial T_1}{\partial x} \frac{\eta P+RT_w}{R\theta_1+\eta P+RT_w} \frac{\delta_1}{A_1^2}$$

$$- \frac{2}{\pi\sqrt{\pi}} \theta_1^2 \mu_1 \frac{\partial \mu_1}{\partial t} \frac{R \Omega_1 \delta_1^2}{R\theta_1+\eta P+RT_w} \Big[\frac{1}{\sqrt{2}\,A_1^2} \tan^{-1}\left(\frac{B_1}{\sqrt{2}\,A_1}\right) + \frac{B_1}{A_1^2(A_1^2+B_1^2)^{1/2}} \tan^{-1}\left(\frac{A_1}{(A_1^2+B_1^2)^{1/2}}\right) \Big]$$

$$+\frac{B_2}{(A_1^2+A_2^2)(A_1^2+A_2^2+B_2^2)}\Bigg]-\frac{\Omega_1\delta_1}{\sqrt{\pi}}C_p\,\theta_1\mu_1\frac{\partial T_w}{\partial x}\frac{B_1}{A_1^2(A_1^2+B_1^2)^{1/2}}-\frac{\Omega_2\delta_1}{\sqrt{\pi}}C_p\,\theta_1\mu_1\frac{\partial T_w}{\partial x}\frac{B_2}{A_1^2(A_1^2+B_2^2}$$

$$+\frac{B_1}{A_1^2(B_1^2+2A_1^2)}\Bigg]+\frac{2}{\pi\sqrt{\pi}}\Omega_1\delta_1\delta_2C_p\,\theta_1^2\frac{\partial\mu_1}{\partial x}A_2\left[\frac{1}{(A_1^2+A_2^2)^{3/2}}\tan^{-1}\left(\frac{B_1}{(A_1^2+A_2^2)^{1/2}}\right)+\frac{}{(A_1^2}\right.$$

$$+\frac{2}{\pi\sqrt{\pi}}\Omega_2\delta_1^2C_p\,\theta_1^2\frac{\partial\mu_1}{\partial x}A_1\left[\frac{1}{\sqrt{2}\,A_1^3}\tan^{-1}\left(\frac{B_2}{\sqrt{2}\,A_1}\right)+\frac{B_2}{A_1^2(B_2^2+2A_1^2)}\right]+\frac{2}{\pi\sqrt{\pi}}\Omega_2\delta_1\delta_2C_p\,\theta_1^2\frac{\partial\mu}{\partial x}$$

$$+\frac{B_2}{(A_2^2+A_1^2)(B_2^2+A_1^2+A_2^2)}\Bigg]+\frac{4}{\pi\sqrt{\pi}}C_p\,\theta_1^2\mu_1\frac{\partial B_1}{\partial x}A_1\frac{\Omega_1\delta_1^2}{(2A_1^2+B_1^2)^2}+\frac{4}{\pi\sqrt{\pi}}C_p\,\theta_1^2\mu_1\frac{\partial B_1}{\partial x}A_2\frac{\Omega_1\delta_1\delta}{(A_1^2+A_2^2}$$

$$+\frac{2}{\pi}C_p\,\theta_1^2\frac{\partial\mu_1}{\partial x}\frac{A_2}{B_1}\frac{\Omega_1\delta_1\delta_2}{(A_1^2+A_2^2+B_1^2)^{1/2}}-\frac{2}{\pi\sqrt{\pi}}C_p\,\theta_1^2\mu_1\frac{\partial B_1}{\partial x}\frac{A_1}{B_1^2}\frac{\Omega_1\delta_1^2}{(2A_1^2+B_1^2)}-\frac{2}{\pi\sqrt{\pi}}C_p\,\theta_1^2\mu_1\frac{\partial B_1}{\partial x}\frac{A}{B}$$

$$-\frac{4}{\pi\sqrt{\pi}}C_p\,\theta_1^2\mu_1A_1\frac{\partial B_1}{\partial x}\frac{\Omega_1\delta_1^2}{(2A_1^2+B_1^2)^2}-\frac{4}{\pi\sqrt{\pi}}C_p\,\theta_1^2\mu_1A_2\frac{\partial B_1}{\partial x}\frac{\Omega_1\delta_1\delta_2}{(A_1^2+A_2^2+B_1^2)^2}+\frac{4}{\pi\sqrt{\pi}}C_p\,\theta_1^2\mu_1$$

$$+\frac{4}{\pi\sqrt{\pi}}C_p\,\theta_1^2\mu_1A_2\frac{\partial B_2}{\partial x}\frac{\Omega_2\delta_1\delta_2}{(A_1^2+A_2^2+B_2^2)^2}+\frac{2}{\pi\sqrt{\pi}}C_p\,\theta_1^2\frac{\partial\mu_1}{\partial x}\frac{A_1}{B_2}\frac{\Omega_2\delta_1^2}{(2A_1^2+B_2^2)}+\frac{2}{\pi\sqrt{\pi}}C_p\,\theta_1^2\frac{\partial\mu_1}{\partial x}$$

$$-\frac{2}{\pi\sqrt{\pi}}C_p\,\theta_1^2\mu_1\frac{\partial B_2}{\partial x}\frac{A_1}{B_2^2}\frac{\Omega_2\delta_1^2}{(2A_1^2+B_2^2)}-\frac{2}{\pi\sqrt{\pi}}C_p\,\theta_1^2\mu_1\frac{\partial B_2}{\partial x}\frac{A_2}{B_2^2}\frac{\Omega_2\delta_1\delta_2}{(A_1^2+A_2^2+B_2^2)}-\frac{4}{\pi\sqrt{\pi}}C_p\,\theta_1^2\mu_1A_1$$

$$-\frac{4}{\pi\sqrt{\pi}}C_p\,\theta_1^2\mu_1A_2\frac{\partial B_2}{\partial x}\frac{\Omega_2\delta_1\delta_2}{(A_1^2+A_2^2+B_2^2)^2}-\frac{1}{\pi\sqrt{\pi}}C_p\,\theta_1^2\frac{\partial\mu_1}{\partial x}\frac{\Omega_1\delta_1^2}{A_1B_1}-\frac{2}{\pi\sqrt{\pi}}C_p\,\theta_1^2\frac{\partial\mu_1}{\partial x}\frac{A_2}{B_1}\frac{\Omega_1}{(A_1^2+}$$

$$+\frac{2}{\pi\sqrt{\pi}}C_p\,\theta_1^2\mu_1\frac{\partial B_1}{\partial x}\frac{A_2}{B_1^2}\frac{\Omega_1\delta_1\delta_2}{(A_1^2+A_2^2)}-\frac{\Omega_2\delta_1^2}{\pi\sqrt{\pi}}C_p\,\theta_1^2\frac{\partial\mu_1}{\partial x}\frac{1}{A_1B_2}-\frac{2}{\pi\sqrt{\pi}}C_p\,\theta_1^2\frac{\partial\mu_1}{\partial x}\frac{A_2}{B_2}\frac{\Omega_2\delta_1\delta_2}{(A_1^2+A_2^2)}$$

$$+\frac{2}{\pi\sqrt{\pi}}C_p\,\theta_1^2\mu_1\frac{\partial B_2}{\partial x}\frac{A_2}{B_2^2}\frac{\Omega_2\delta_1\delta_2}{(A_1^2+A_2^2)}-\frac{1}{2}\sqrt{\frac{2}{\pi}}C_p\frac{CT_1}{PR}\theta_1^2\frac{\rho_1}{\rho_0^2}\delta_1^2-\frac{2}{\sqrt{\pi}}\frac{C_p\,\theta_1^2A_2^3\rho CT_1\delta_1\delta_2}{\rho_0^2P_r(A_1^2+A_2^2)(A_1^2+A_2^2)^{1/2}}$$

$$+\frac{8}{\pi\sqrt{\pi}}\frac{\Omega_1\Omega_2\delta_1\theta_1\mu_1^2B_1B_2CT_1}{(A_1^2+B_1^2+B_2^2)}\frac{\rho_1}{\rho_0^2}+\frac{4}{\pi\sqrt{\pi}}\frac{\Omega_2^2\delta_1\theta_1\mu_1^2B_2^2CT_1}{(A_1^2+2B_2^2)}\frac{\rho_1}{\rho_0^2}+\frac{1}{2}\sqrt{\frac{2}{\pi}}\frac{\theta_1^2C_pR}{R\theta_1+\eta P+RT}$$

$$+\frac{1}{\sqrt{\pi}}C_p\,\theta_1^2\frac{\partial T_1}{\partial t}\frac{A_2}{A_1^2(A_1^2+A_2^2)^{1/2}}\frac{R\delta_1\delta_2}{R\theta_1+\eta P+RT_w}+\frac{1}{\sqrt{\pi}}C_p\,\theta_1\frac{\partial T_1}{\partial t}\frac{\eta P+RT_w}{R\theta_1+\eta P+RT_w}\frac{\delta_1}{A_1^2}+\frac{1}{2}\sqrt{\frac{2}{\pi}}C$$

$$+\frac{1}{\sqrt{\pi}}C_p\,\theta_1^2\mu_1\frac{\partial T_1}{\partial x}\frac{R\delta_1\delta_2}{R\theta_1+\eta P+RT_w}\frac{A_2}{A_1^2(A_1^2+A_2^2)^{1/2}}+\frac{1}{\sqrt{\pi}}C_p\,\theta_1\mu_1\frac{\partial T_1}{\partial x}\frac{\eta P+RT_w}{R\theta_1+\eta P+RT_w}\frac{\delta_1}{A_1^2}$$

$$-\frac{2}{\pi\sqrt{\pi}}\theta_1^2\mu_1\frac{\partial\mu_1}{\partial t}\frac{R\Omega_1\delta_1^2}{R\theta_1+\eta P+RT_w}\left[\frac{1}{\sqrt{2}\,A_1^2}\tan^{-1}\left(\frac{B_1}{\sqrt{2}\,A_1}\right)+\frac{B_1}{A_1^2(A_1^2+B_1^2)^{1/2}}\tan^{-1}\left(\frac{A_1}{(A_1^2+B_1^2)^{1/2}}\right)\right.$$

$$\frac{\partial T_w}{\partial x}\frac{B_1}{A_1^2(A_1^2+B_1^2)^{1/2}} - \frac{\Omega_2\delta_1}{\sqrt{\pi}} C_p\,\theta_1\,\mu_1\frac{\partial T_w}{\partial x}\frac{B_2}{A_1^2(A_1^2+B_2^2)^{1/2}} + \frac{2}{\pi\sqrt{\pi}}\Omega_1\delta_1^2 C_p\theta_1^2\frac{\partial\mu_1}{\partial x}A_1\left[\frac{1}{\sqrt{2}A_1^3}\tan^{-1}\left(\frac{B_1}{\sqrt{2}A_1}\right)\right.$$

$$\frac{\partial\mu_1}{\partial x}A_2\left[\frac{1}{(A_1^2+A_2^2)^{3/2}}\tan^{-1}\left(\frac{B_1}{(A_1^2+A_2^2)^{1/2}}\right)+\frac{B_1}{(A_1^2+A_2^2)(B_1^2+A_1^2+A_2^2)}\right]$$

$$\left.\frac{B_2}{A_1}\right)+\frac{E_2}{A_1^2(B_2^2+2A_1^2)}\Bigg] + \frac{2}{\pi\sqrt{\pi}}\Omega_2\delta_1\delta_2 C_p\theta_1^2\frac{\partial\mu_1}{\partial x}A_2\left[\frac{1}{(A_1^2+A_2^2)^{3/2}}\tan^{-1}\left(\frac{B_2}{(A_1^2+A_2^2)^{1/2}}\right)\right.$$

$$\frac{B_1}{x}A_1\frac{\Omega_1\delta_1^2}{(2A_1^2+B_1^2)^2} + \frac{4}{\pi\sqrt{\pi}}C_p\theta_1^2\mu_1\frac{\partial B_1}{\partial x}A_2\frac{\Omega_1\delta_1\delta_2}{(A_1^2+A_2^2+B_1^2)^2} + \frac{2}{\pi\sqrt{\pi}}C_p\theta_1^2\frac{\partial\mu_1}{\partial x}\frac{A_1}{B_1}\frac{\Omega_1\delta_1^2}{(2A_1^2+B_1^2)}$$

$$C_p\theta_1^2\mu_1\frac{\partial B_1}{\partial x}\frac{A_1}{B_1^2}\frac{\Omega_1\delta_1^2}{(2A_1^2+B_1^2)} - \frac{2}{\pi\sqrt{\pi}}C_p\theta_1^2\mu_1\frac{\partial B_1}{\partial x}\frac{A_2}{B_1^2}\frac{\Omega_1\delta_1\delta_2}{(A_1^2+A_2^2+B_1^2)}$$

$$\overline{\pi}\; C_p\theta_1^2\mu_1 A_2\frac{\partial B_1}{\partial x}\frac{\Omega_1\delta_1\delta_2}{(A_1^2+A_2^2+B_1^2)^2} + \frac{4}{\pi\sqrt{\pi}}C_p\theta_1^2\mu_1 A_1\frac{\partial B_2}{\partial x}\frac{\Omega_2\delta_1^2}{(2A_1^2+B_2^2)^2}$$

$$\frac{2}{\sqrt{\pi}}C_p\theta_1^2\frac{\partial\mu_1}{\partial x}\frac{A_1}{B_2}\frac{\Omega_2\delta_1^2}{(2A_1^2+B_2^2)} + \frac{2}{\pi\sqrt{\pi}}C_p\theta_1^2\frac{\partial\mu_1}{\partial x}\frac{A_2}{B_2}\frac{\Omega_2\delta_1\delta_2}{(A_1^2+A_2^2+B_2^2)}$$

$$C_p\theta_1^2\mu_1\frac{\partial B_2}{\partial x}\frac{A_2}{B_2^2}\frac{\Omega_2\delta_1\delta_2}{(A_1^2+A_2^2+B_2^2)} - \frac{4}{\pi\sqrt{\pi}}C_p\theta_1^2\mu_1 A_1\frac{\partial B_2}{\partial x}\frac{\Omega_1\delta_1^2}{(2A_1^2+B_2^2)^2}$$

$$\frac{1}{\pi\sqrt{\pi}}C_p\theta_1^2\frac{\partial\mu_1}{\partial x}\frac{\Omega_1\delta_1^2}{A_1B_1} - \frac{2}{\pi\sqrt{\pi}}C_p\theta_1^2\frac{\partial\mu_1}{\partial x}\frac{A_2}{B_1}\frac{\Omega_1\delta_1\delta_2}{(A_1^2+A_2^2)} + \frac{1}{\pi\sqrt{\pi}}C_p\theta_1^2\mu_1\frac{\partial B_1}{\partial x}\frac{\Omega_1\delta_1^2}{A_1B_1^2}$$

$$C_p\theta_1^2\frac{\partial\mu_1}{\partial x}\frac{1}{A_1B_2} - \frac{2}{\pi\sqrt{\pi}}C_p\theta_1^2\frac{\partial\mu_1}{\partial x}\frac{A_2}{B_2}\frac{\Omega_2\delta_1\delta_2}{(A_1^2+A_2^2)} + \frac{1}{\pi\sqrt{\pi}}C_p\theta_1^2\mu_1\frac{\partial B_2}{\partial x}\frac{\Omega_2\delta_1^2}{A_1B_2^2}$$

$$C_p\frac{CT_1}{PR}\theta_1^2\frac{\rho_1}{\rho_0^2}\delta_1^2 - \frac{2}{\sqrt{\pi}}\frac{C_p\theta_1^2A_2^3\rho CT_1\delta_1\delta_2}{\rho_0^2P_r(A_1^2+A_2^2)(A_1^2+A_2^2)^{1/2}} + \frac{4}{\pi\sqrt{\pi}}\frac{\Omega_1^2\delta_1\theta_1\mu_1^2B_1^2CT_1}{(A_1^2+2B_1^2)}\frac{\rho_1}{\rho_0^2}$$

$$\frac{\Omega_2^2\delta_1\theta_1\mu_1^2B_2^2CT_1}{(A_1^2+2B_2^2)}\frac{\rho_1}{\rho_0^2} + \frac{1}{2}\sqrt{\frac{2}{\pi}}\frac{\theta_1^2C_pR}{R\theta_1+\eta P+RT_w}\frac{\partial T_1}{\partial t}\frac{\delta_1^2}{A_1^2}$$

$$+\frac{1}{\sqrt{\pi}}C_p\theta_1\frac{\partial T_1}{\partial t}\frac{\eta P+RT_w}{R\theta_1+\eta P+RT_w}\frac{\delta_1}{A_1^2} + \frac{1}{2}\sqrt{\frac{2}{\pi}}C_p\theta_1^2\mu_1\frac{\partial T_1}{\partial x}\frac{R}{R\theta_1+\eta P+RT_w}\frac{\delta_1^2}{A_1^2}$$

$$A_2^2)^{1/2} + \frac{1}{\sqrt{\pi}}C_p\theta_1\mu_1\frac{\partial T_1}{\partial x}\frac{\eta P+RT_w}{R\theta_1+\eta P+RT_w}\frac{\delta_1}{A_1^2}$$

$$n^{-1}\left(\frac{B_1}{\sqrt{2}A_1}\right)+\frac{B_1}{A_1^2(A_1^2+B_1^2)^{1/2}}\tan^{-1}\left(\frac{A_1}{(A_1^2+B_1^2)^{1/2}}\right)\Bigg]\Bigg]$$

$$- \frac{2}{\pi\sqrt{\pi}} \theta_1^2 \mu_1 \frac{\partial \mu_1}{\partial t} \frac{R\Omega_1 \delta_1 \delta_2}{R\theta_1 + \eta P + RT_w} \left[\frac{1}{\sqrt{2}A_2^2} \tan^{-1}\left(\frac{B_1}{\sqrt{2}A_2}\right) + \frac{B_1}{A_2^2(A_2^2+B_1^2)^{1/2}} \tan^{-1}\left(\frac{A_2}{(A_2^2+B_1^2)^{1/2}}\right)\right] - \frac{1}{\sqrt{\pi}} \theta_1 \mu_1 \frac{\partial \mu_1}{\partial t} \frac{\eta P + RT_w}{R\theta_1 + \eta P + RT_w} \frac{B_1 \Omega_1 \delta_1}{A_1^2(A_1^2+B_1^2)^{1/2}}$$

$$- \frac{2}{\pi\sqrt{\pi}} \theta_1^2 \mu_1 \frac{\partial \mu_1}{\partial t} \frac{R\Omega_2 \delta_1^2}{R\theta_1 + \eta P + RT_w} \left[\frac{1}{\sqrt{2}A_1^2} \tan^{-1}\left(\frac{B_2}{\sqrt{2}A_1}\right) + \frac{B_2}{A_1^2(A_1^2+B_2^2)^{1/2}} \tan^{-1}\left(\frac{A_1}{(A_1^2+B_2^2)^{1/2}}\right)\right]$$

$$- \frac{2}{\pi\sqrt{\pi}} \theta_1^2 \mu_1 \frac{\partial \mu_1}{\partial t} \frac{R\Omega_2 \delta_1 \delta_2}{R\theta_1 + \eta P + RT_w} \left[\frac{1}{\sqrt{2}A_2^2} \tan^{-1}\left(\frac{B_2}{\sqrt{2}A_2}\right) + \frac{B_2}{A_2^2(A_2^2+B_2^2)^{1/2}} \tan^{-1}\left(\frac{A_2}{(A_2^2+B_2^2)^{1/2}}\right)\right]$$

$$- \frac{1}{\sqrt{\pi}} \theta_1 \mu_1 \frac{\partial \mu_1}{\partial t} \frac{\eta P + RT_w}{R\theta_1 + \eta P + RT_w} \frac{B_2 \Omega_2 \delta_1}{A_1^2(A_1^2+B_2^2)^{1/2}} - \frac{2}{\pi\sqrt{\pi}} \theta_1^2 \mu_1^2 \frac{\partial \mu_1}{\partial x} \frac{R\Omega_1 \delta_1^2}{R\theta_1 + \eta P + RT_w} \left[\frac{1}{\sqrt{2}A_1^2} \tan^{-1}\left(\frac{B_1}{\sqrt{2}A_1}\right) + \frac{B_1}{A_1^2(A_1^2+B_1^2)^{1/2}} \tan^{-1}\left(\frac{A_1}{(A_1^2+B_1^2)^{1/2}}\right)\right]$$

$$- \frac{2}{\pi\sqrt{\pi}} \theta_1^2 \mu_1^2 \frac{\partial \mu_1}{\partial x} \frac{R\Omega_1 \delta_1 \delta_2}{R\theta_1 + \eta P + RT_w} \left[\frac{1}{\sqrt{2}A_2^2} \tan^{-1}\left(\frac{B_1}{\sqrt{2}A_2}\right) + \frac{B_1}{A_2^2(A_2^2+B_1^2)^{1/2}} \tan^{-1}\left(\frac{A_2}{(A_2^2+B_1^2)^{1/2}}\right)\right] - \frac{1}{\sqrt{\pi}} \theta_1 \mu_1^2 \frac{\partial \mu_1}{\partial x} \frac{\eta P + RT_w}{R\theta_1 + \eta P + RT_w} \frac{B_1 \Omega_1 \delta_1}{A_1^2(A_1^2+B_1^2)^{1/2}}$$

$$- \frac{2}{\pi\sqrt{\pi}} \theta_1^2 \mu_1^2 \frac{\partial \mu_1}{\partial x} \frac{R\Omega_2 \delta_1^2}{R\theta_1 + \eta P + RT_w} \left[\frac{1}{\sqrt{2}A_1^2} \tan^{-1}\left(\frac{B_2}{\sqrt{2}A_1}\right) + \frac{B_2}{A_1^2(A_1^2+B_2^2)^{1/2}} \tan^{-1}\left(\frac{A_1}{(A_1^2+B_2^2)^{1/2}}\right)\right]$$

$$- \frac{2}{\pi\sqrt{\pi}} \theta_1^2 \mu_1^2 \frac{\partial \mu_1}{\partial x} \frac{R\Omega_2 \delta_1 \delta_2}{R\theta_1 + \eta P + RT_w} \left[\frac{1}{\sqrt{2}A_2^2} \tan^{-1}\left(\frac{B_2}{\sqrt{2}A_2}\right) + \frac{B_2}{A_2^2(A_2^2+B_2^2)^{1/2}} \tan^{-1}\left(\frac{A_2}{(A_2^2+B_2^2)^{1/2}}\right)\right] - \frac{1}{\sqrt{\pi}} \theta_1 \mu_1^2 \frac{\partial \mu_1}{\partial x} \frac{\eta P + RT_w}{R\theta_1 + \eta P + RT_w} \frac{B_2 \Omega_2 \delta_1}{A_1^2(A_1^2+B_2^2)^{1/2}}$$

$$+ \frac{1}{2}\sqrt{\frac{2}{\pi}} \theta_1^2 \mu_1 \frac{\partial \mu_1}{\partial t} \frac{R}{R\theta_1 + \eta P + RT_w} \frac{\delta_1^2}{A_1^2} + \frac{1}{\sqrt{\pi}} \theta_1^2 \mu_1 \frac{\partial \mu_1}{\partial t} \frac{R}{R\theta_1 + \eta P + RT_w} \frac{A_2 \delta_1 \delta_2}{A_1^2(A_1^2+A_2^2)^{1/2}} + \frac{1}{\sqrt{\pi}} \theta_1 \mu_1 \frac{\partial \mu_1}{\partial t} \frac{\eta P + RT_w}{R\theta_1 + \eta P + RT_w} \frac{\delta_1}{A_1^2}$$

$$+ \frac{1}{2}\sqrt{\frac{2}{\pi}} \theta_1^2 \mu_1^2 \frac{\partial \mu_1}{\partial x} \frac{R}{R\theta_1 + \eta P + RT_w} \frac{\delta_1^2}{A_1^2} + \frac{1}{\sqrt{\pi}} \theta_1^2 \mu_1^2 \frac{\partial \mu_1}{\partial x} \frac{R\delta_1\delta_2}{R\theta_1 + \eta P + RT_w} \frac{A_2}{A_1^2(A_1^2+A_2^2)^{1/2}}$$

$$+ \frac{1}{\sqrt{\pi}} \theta_1 \mu_1^2 \frac{\partial \mu_1}{\partial x} \frac{\eta P + RT_w}{R\theta_1 + \eta P + RT_w} \frac{\delta_1}{A_1^2} \Bigg\} \Bigg/ \frac{1}{4}\sqrt{\frac{2}{\pi}}\, \delta_1^2 C_p \frac{\theta_1^2}{A_1^3} \qquad [3.28]$$

$$\frac{\partial A_2}{\partial t} = \Bigg\{ - \frac{1}{\sqrt{\pi}} \delta_1 \delta_2 C_p \theta_1 \frac{\partial \theta_1}{\partial t} \frac{A_1}{A_2^2(A_1^2+A_2^2)^{1/2}} - \frac{1}{2}\sqrt{\frac{2}{\pi}} \delta_2^2 C_p \theta_1 \frac{\partial \theta_1}{\partial t} \frac{1}{A_2^2} - \frac{1}{\sqrt{\pi}} C_p \theta_1^2 \frac{\partial A_1}{\partial t} \frac{\delta_1 \delta_2}{(A_1^2+A_2^2)^{1/2}(A_1^2+A_2^2)} - \frac{1}{\sqrt{\pi}} \delta_2 C_p \theta_1 \frac{\partial T_w}{\partial t} \frac{1}{A_2^2}$$

$$- \frac{2}{\pi\sqrt{\pi}} \Omega_1 \delta_1 \delta_2 C_p \theta_1 \frac{\partial \theta_1}{\partial x} \mu_1 \left[\frac{A_1}{A_2^2(A_1^2+A_2^2)^{1/2}} \tan^{-1}\left(\frac{B_1}{(A_1^2+A_2^2)^{1/2}}\right) + \frac{B_1}{A_2^2(A_2^2+B_1^2)^{1/2}} \tan^{-1}\left(\frac{A_1}{(A_2^2+B_1^2)^{1/2}}\right)\right]$$

$$- \frac{2}{\pi\sqrt{\pi}} \Omega_1 \delta_2^2 C_p \theta_1 \frac{\partial \theta_1}{\partial x} \mu_1 \left[\frac{1}{\sqrt{2}A_2^2} \tan^{-1}\left(\frac{B_1}{\sqrt{2}A_2}\right) + \frac{B_1}{A_2^2(A_2^2+B_1^2)^{1/2}} \tan^{-1}\left(\frac{A_2}{(A_2^2+B_1^2)^{1/2}}\right)\right]$$

$$- \frac{2}{\pi\sqrt{\pi}} \Omega_1 \delta_1 \delta_2 C_p \theta_1^2 \mu_1 \frac{\partial A_1}{\partial x} \left[\frac{1}{(A_1^2+A_2^2)^{3/2}} \tan^{-1}\left(\frac{B_1}{(A_1^2+A_2^2)^{1/2}}\right) + \frac{B_1}{(A_1^2+A_2^2)(A_1^2+A_2^2+B_1^2)}\right] - \frac{2}{\pi\sqrt{\pi}} \Omega_1 \delta_2^2 C_p \theta_1^2 \mu_1 \frac{\partial A_2}{\partial x} \left[\frac{1}{\sqrt{2}A_2^2} \tan^{-1}\left(\frac{B_1}{\sqrt{2}A_2}\right)$$

$$- \frac{2}{\pi\sqrt{\pi}}\,\theta_1^2 \mu_1 \frac{\partial \mu_1}{\partial t} \frac{R\Omega_1 \delta_1 \delta_2}{R\theta_1 + \eta P + RT_w} \left[\frac{1}{\sqrt{2}\,A_2^2} \tan^{-1}\left(\frac{B_1}{\sqrt{2}\,A_2}\right) + \frac{B_1}{A_2^2 (A_2^2 + B_1^2)^{1/2}} \tan^{-1}\left(\frac{A_2}{(A_2^2 + B_1^2)^{1/2}}\right)\right.$$

$$- \frac{2}{\pi\sqrt{\pi}}\,\theta_1^2 \mu_1 \frac{\partial \mu_1}{\partial t} \frac{R\Omega_2 \delta_1^2}{R\theta_1 + \eta P + RT_w} \left[\frac{1}{\sqrt{2}\,A_1^2} \tan^{-1}\left(\frac{B_2}{\sqrt{2}\,A_1}\right) + \frac{B_2}{A_1^2 (A_1^2 + B_2^2)^{1/2}} \tan^{-1}\left(\frac{A_1}{(A_1^2 + B_2^2)^{1/2}}\right.\right.$$

$$- \frac{2}{\pi\sqrt{\pi}}\,\theta_1^2 \mu_1 \frac{\partial \mu_1}{\partial t} \frac{R\Omega_2 \delta_1 \delta_2}{R\theta_1 + \eta P + RT_w} \left[\frac{1}{\sqrt{2}\,A_2^2} \tan^{-1}\left(\frac{B_2}{\sqrt{2}\,A_2}\right) + \frac{B_2}{A_2^2 (A_2^2 + B_2^2)^{1/2}} \tan^{-1}\left(\frac{A_2}{(A_2^2 + B_2^2)^{1/2}}\right.\right.$$

$$- \frac{1}{\sqrt{\pi}}\,\theta_1 \mu_1 \frac{\partial \mu_1}{\partial t} \frac{\eta P + RT_w}{R\theta_1 + \eta P + RT_w} \frac{B_2 \Omega_2 \delta_1}{A_1^2 (A_1^2 + B_2^2)^{1/2}} - \frac{2}{\pi\sqrt{\pi}}\,\theta_1^2 \mu_1^2 \frac{\partial \mu_1}{\partial x} \frac{R\Omega_1 \delta_1^2}{R\theta_1 + \eta P + RT_w} \left[\frac{1}{\sqrt{2}\,A_1^2} \tan^{-1}\left(\right.\right.$$

$$- \frac{2}{\pi\sqrt{\pi}}\,\theta_1^2 \mu_1^2 \frac{\partial \mu_1}{\partial x} \frac{R\Omega_1 \delta_1 \delta_2}{R\theta_1 + \eta P + RT_w} \left[\frac{1}{\sqrt{2}\,A_2^2} \tan^{-1}\left(\frac{B_1}{\sqrt{2}\,A_2}\right) + \frac{B_1}{A_2^2 (A_2^2 + B_1^2)^{1/2}} \tan^{-1}\left(\frac{A_2}{(A_2^2 + B_1^2)^{1/2}}\right.\right.$$

$$- \frac{2}{\pi\sqrt{\pi}}\,\theta_1^2 \mu_1^2 \frac{\partial \mu_1}{\partial x} \frac{R\Omega_2 \delta_1^2}{R\theta_1 + \eta P + RT_w} \left[\frac{1}{\sqrt{2}\,A_1^2} \tan^{-1}\left(\frac{B_2}{\sqrt{2}\,A_1}\right) + \frac{B_2}{A_1^2 (A_1^2 + B_2^2)^{1/2}} \tan^{-1}\left(\frac{A_1}{(A_1^2 + B_2^2)^{1/2}}\right.\right.$$

$$- \frac{2}{\pi\sqrt{\pi}}\,\theta_1^2 \mu_1^2 \frac{\partial \mu_1}{\partial x} \frac{R\Omega_2 \delta_1 \delta_2}{R\theta_1 + \eta P + RT_w} \left[\frac{1}{\sqrt{2}\,A_2^2} \tan^{-1}\left(\frac{B_2}{\sqrt{2}\,A_2}\right) + \frac{B_2}{A_2^2 (A_2^2 + B_2^2)^{1/2}} \tan^{-1}\left(\frac{A_2}{(A_2^2 + B_2^2)^{1/2}}\right.\right.$$

$$+ \frac{1}{2}\sqrt{\frac{2}{\pi}}\,\theta_1^2 \mu_1 \frac{\partial \mu_1}{\partial t} \frac{R}{R\theta_1 + \eta P + RT_w} \frac{\delta_1^2}{A_1^2} + \frac{1}{\sqrt{\pi}}\,\theta_1^2 \mu_1 \frac{\partial \mu_1}{\partial t} \frac{R}{R\theta_1 + \eta P + RT_w} \frac{A_2 \delta_1 \delta_2}{A_1^2 (A_1^2 + A_2^2)^{1/2}} +$$

$$+ \frac{1}{2}\sqrt{\frac{2}{\pi}}\,\theta_1^2 \mu_1^2 \frac{\partial \mu_1}{\partial x} \frac{R}{R\theta_1 + \eta P + RT_w} \frac{\delta_1^2}{A_1^2} + \frac{1}{\sqrt{\pi}}\,\theta_1^2 \mu_1^2 \frac{\partial \mu_1}{\partial x} \frac{R\,\delta_1 \delta_2}{R\theta_1 + \eta P + RT_w} \frac{A_2}{A_1^2 (A_1^2 + A_2^2)^{1/2}}$$

$$+ \frac{1}{\sqrt{\pi}}\,\theta_1 \mu_1^2 \frac{\partial \mu_1}{\partial x} \frac{\eta P + RT_w}{R\theta_1 + \eta P + RT_w} \frac{\delta_1}{A_1^2} \Big\} \Big/ \frac{1}{4}\sqrt{\frac{2}{\pi}}\;\delta_1^2 C_p \frac{\theta_1^2}{A_1^3}$$

$$\frac{\partial A_2}{\partial t} = \Big\{ - \frac{1}{\sqrt{\pi}}\,\delta_1 \delta_2 C_p \theta_1 \frac{\partial \theta_1}{\partial t} \frac{A_1}{A_2^2 (A_1^2 + A_2^2)^{1/2}} - \frac{1}{2}\sqrt{\frac{2}{\pi}}\;\delta_2^2 C_p \theta_1 \frac{\partial \theta_1}{\partial t} \frac{1}{A_2^2} - \frac{1}{\sqrt{\pi}} C_p \theta_1^2 \frac{\partial A_1}{\partial t} \frac{\delta_1 \delta_2}{(A_1^2 + A_2^2)^{1/2} (A_1^2 +}$$

$$- \frac{2}{\pi\sqrt{\pi}}\,\Omega_1 \delta_1 \delta_2 C_p \theta_1 \frac{\partial \theta_1}{\partial x} \mu_1 \left[\frac{A_1}{A_2^2 (A_1^2 + A_2^2)^{1/2}} \tan^{-1}\left(\frac{B_1}{(A_1^2 + A_2^2)^{1/2}}\right) + \frac{B_1}{A_2^2 (A_2^2 + B_1^2)^{1/2}} \tan^{-1}\left(\frac{}{(A_2^2 +}\right.\right.$$

$$- \frac{2}{\pi\sqrt{\pi}}\,\Omega_1 \delta_2^2 C_p \theta_1 \frac{\partial \theta_1}{\partial x} \mu_1 \left[\frac{1}{\sqrt{2}\,A_2^2} \tan^{-1}\left(\frac{B_1}{\sqrt{2}\,A_2}\right) + \frac{B_1}{A_2^2 (A_2^2 + B_1^2)^{1/2}} \tan^{-1}\left(\frac{A_2}{(A_2^2 + B_1^2)^{1/2}}\right)\right]$$

$$- \frac{2}{\pi\sqrt{\pi}}\,\Omega_1 \delta_1 \delta_2 C_p \theta_1^2 \mu_1 \frac{\partial A_1}{\partial x} \left[\frac{1}{(A_1^2 + A_2^2)^{3/2}} \tan^{-1}\left(\frac{B_1}{(A_1^2 + A_2^2)^{1/2}}\right) + \frac{B_1}{(A_1^2 + A_2^2)(A_1^2 + A_2^2 + B_1^2)}\right] - \frac{2}{\pi\sqrt{\pi}}$$

$$\tan^{-1}\left(\frac{B_1}{\sqrt{2}\,A_2}\right) + \frac{B_1}{A_2^2(A_2^2+B_1^2)^{1/2}} \tan^{-1}\left(\frac{A_2}{(A_2^2+B_1^2)^{1/2}}\right)\Big] - \frac{1}{\sqrt{\pi}}\theta_1 \mu_1 \frac{\partial \mu_1}{\partial t} \frac{\eta P + R T_w}{R\theta_1 + \eta P + R T_w} \frac{B_1 \Omega_1 \delta_1}{A_1^2(A_1^2+B_1^2)^{1/2}}$$

$$\tan^{-1}\left(\frac{B_2}{\sqrt{2}\,A_1}\right) + \frac{B_2}{A_1^2(A_1^2+B_2^2)^{1/2}} \tan^{-1}\left(\frac{A_1}{(A_1^2+B_2^2)^{1/2}}\right)\Big]$$

$$\tan^{-1}\left(\frac{B_2}{\sqrt{2}\,A_2}\right) + \frac{B_2}{A_2^2(A_2^2+B_2^2)^{1/2}} \tan^{-1}\left(\frac{A_2}{(A_2^2+B_2^2)^{1/2}}\right)\Big]$$

$$- \frac{2}{\pi\sqrt{\pi}}\theta_1^2 \mu_1^2 \frac{\partial \mu_1}{\partial x} \frac{R\Omega_1 \delta_1^2}{R\theta_1 + \eta P + R T_w} \Big[\frac{1}{\sqrt{2}\,A_1^2} \tan^{-1}\left(\frac{B_1}{\sqrt{2}\,A_1}\right) + \frac{B_1}{A_1^2(A_1^2+B_1^2)^{1/2}} \tan^{-1}\left(\frac{A_1}{(A_1^2+B_1^2)^{1/2}}\right)\Big]$$

$$\tan^{-1}\left(\frac{B_1}{\sqrt{2}\,A_2}\right) + \frac{B_1}{A_2^2(A_2^2+B_1^2)^{1/2}} \tan^{-1}\left(\frac{A_2}{(A_2^2+B_1^2)^{1/2}}\right)\Big] - \frac{1}{\sqrt{\pi}}\theta_1 \mu_1^2 \frac{\partial \mu_1}{\partial x} \frac{\eta P + R T_w}{R\theta_1 + \eta P + R T_w} \frac{B_1 \Omega_1 \delta_1}{A_1^2(A_1^2+B_1^2)^{1/2}}$$

$$\tan^{-1}\left(\frac{B_2}{\sqrt{2}\,A_1}\right) + \frac{B_2}{A_1^2(A_1^2+B_2^2)^{1/2}} \tan^{-1}\left(\frac{A_1}{(A_1^2+B_2^2)^{1/2}}\right)\Big]$$

$$\tan^{-1}\left(\frac{B_2}{\sqrt{2}\,A_2}\right) + \frac{B_2}{A_2^2(A_2^2+B_2^2)^{1/2}} \tan^{-1}\left(\frac{A_2}{(A_2^2+B_2^2)^{1/2}}\right)\Big] - \frac{1}{\sqrt{\pi}}\theta_1 \mu_1^2 \frac{\partial \mu_1}{\partial x} \frac{\eta P + R T_w}{R\theta_1 + \eta P + R T_w} \frac{B_2 \Omega_2 \delta_1}{A_1^2(A_1^2+B_2^2)^{1/2}}$$

$$+ \frac{1}{\sqrt{\pi}}\theta_1^2 \mu_1 \frac{\partial \mu_1}{\partial t} \frac{R}{R\theta_1 + \eta P + R T_w} \frac{A_2 \delta_1 \delta_2}{A_1^2(A_1^2+A_2^2)^{1/2}} + \frac{1}{\sqrt{\pi}}\theta_1 \mu_1 \frac{\partial \mu_1}{\partial t} \frac{\eta P + R T_w}{R\theta_1 + \eta P + R T_w} \frac{\delta_1}{A_1^2}$$

$$\frac{1}{\sqrt{\pi}}\theta_1^2 \mu_1^2 \frac{\partial \mu_1}{\partial x} \frac{R \delta_1 \delta_2}{R\theta_1 + \eta P + R T_w} \frac{A_2}{A_1^2(A_1^2+A_2^2)^{1/2}}$$

$$\sqrt{\frac{2}{\pi}}\ \delta_1^2 C_p \frac{\theta_1^2}{A_1^3} \qquad [3.28]$$

$$\sqrt{\frac{2}{\pi}}\ \delta_2^2 C_p \theta_1 \frac{\partial \theta_1}{\partial t} \frac{1}{A_2^2} - \frac{1}{\sqrt{\pi}} C_p \theta_1^2 \frac{\partial A_1}{\partial t} \frac{\delta_1 \delta_2}{(A_1^2+A_2^2)^{1/2}(A_1^2+A_2^2)} - \frac{1}{\sqrt{\pi}} \delta_2 C_p \theta_1 \frac{\partial T_w}{\partial t} \frac{1}{A_2^2}$$

$$\tan^{-1}\left(\frac{B_1}{(A_1^2+A_2^2)^{1/2}}\right) + \frac{B_1}{A_2^2(A_2^2+B_1^2)^{1/2}} \tan^{-1}\left(\frac{A_1}{(A_2^2+B_1^2)^{1/2}}\right)\Big]$$

$$\frac{B_1}{\sqrt{2}\,A_2}\Big) + \frac{\delta_1}{A_2^2(A_2^2+B_1^2)^{1/2}} \tan^{-1}\left(\frac{A_2}{(A_2^2+B_1^2)^{1/2}}\right)\Big]$$

$$\tan^{-1}\left(\frac{B_1}{(A_1^2+A_2^2)^{1/2}}\right) + \frac{B_1}{(A_1^2+A_2^2)(A_1^2+A_2^2+B_1^2)}\Big] - \frac{2}{\pi\sqrt{\pi}} \Omega_1 \delta_2^2 C_p \theta_1^2 \mu_1 \frac{\partial A_2}{\partial x} \Big[\frac{1}{\sqrt{2}\,A_2^3} \tan^{-1}\left(\frac{B_1}{\sqrt{2}\,A_2}\right)$$

$$+ \frac{B_1}{A_2^2(2A_2^2+B_1^2)} \Big] - \frac{2}{\pi\sqrt{\pi}} \Omega_2 \delta_1 \delta_2 C_p \theta_1 \frac{\partial \theta_1}{\partial x} \mu_1 \left[\frac{A_1}{A_2^2(A_1^2+A_2^2)^{1/2}} \tan^{-1}\left(\frac{B_2}{(A_1^2+A_2^2)^{1/2}}\right) + \frac{B_2}{A_2^2(A_2^2+B_2^2)^{1/2}} \tan^{-1}\left(\frac{A_1}{(A_2^2+B_2^2)^{1/2}}\right) \right]$$

$$- \frac{2}{\pi\sqrt{\pi}} \Omega_2 \delta_2^2 C_p \theta_1 \frac{\partial \theta_1}{\partial x} \mu_1 \left[\frac{1}{\sqrt{2}\, A_2^2} \tan^{-1}\left(\frac{B_2}{\sqrt{2}\, A_2}\right) + \frac{B_2}{A_2^2(A_2^2+B_2^2)^{1/2}} \tan^{-1}\left(\frac{A_2}{(A_2^2+B_2^2)^{1/2}}\right) \right] - \frac{2}{\pi\sqrt{\pi}} \Omega_2 \delta_1 \delta_2 C_p \theta_1^2 \mu_1 \frac{\partial A_1}{\partial x} \Big[\frac{1}{(A_1^2+A_2^2)^{3/2}} \tan^{-1}\left(\frac{B_2}{(A_1^2+A_2^2)^{1/2}}\right)$$

$$+ \frac{B_2}{(A_1^2+A_2^2)(A_1^2+A_2^2+B_2^2)} \Big] - \frac{2}{\pi\sqrt{\pi}} \Omega_2 \delta_2^2 C_p \theta_1 \mu_1 \frac{\partial A_2}{\partial x} \left[\frac{1}{\sqrt{2}\, A_2^2} \tan^{-1}\left(\frac{B_2}{\sqrt{2}\, A_2}\right) + \frac{B_2}{A_2^2(2A_2^2+B_2^2)} \right] - \frac{1}{\sqrt{\pi}} \Omega_1 \delta_2 C_p \theta_1 \mu_1 \frac{\partial T_w}{\partial x} \frac{B_1}{A_2^2(A_2^2+B_1^2)^{1/2}}$$

$$- \frac{1}{\sqrt{\pi}} \Omega_2 \delta_2 C_p \theta_1 \mu_1 \frac{\partial T_w}{\partial x} \frac{B_2}{A_2^2(A_2^2+B_2^2)^{1/2}} + \frac{2}{\pi\sqrt{\pi}} \Omega_1 \delta_1 \delta_2 C_p \theta_1^2 \frac{\partial \mu_1}{\partial x} A_1 \left[\frac{1}{(A_1^2+A_2^2)^{3/2}} \tan^{-1}\left(\frac{B_1}{(A_1^2+A_2^2)^{1/2}}\right) + \frac{B_1}{(A_1^2+A_2^2)(A_1^2+A_2^2+B_1^2)} \right]$$

$$+ \frac{2}{\pi\sqrt{\pi}} \Omega_1 \delta_2^2 C_p \theta_1^2 \frac{\partial \mu_1}{\partial x} A_2 \left[\frac{1}{\sqrt{2}\, A_2^3} \tan^{-1}\left(\frac{B_1}{\sqrt{2}\, A_2}\right) + \frac{B_1}{A_2^2(2A_2^2+B_1^2)} \right] + \frac{2}{\pi\sqrt{\pi}} \Omega_2 \delta_1 \delta_2 C_p \theta_1^2 \frac{\partial \mu_1}{\partial x} A_1 \Big[\frac{1}{(A_1^2+A_2^2)^{3/2}} \tan^{-1}\left(\frac{B_2}{(A_1^2+A_2^2)^{1/2}}\right)$$

$$+ \frac{B_2}{(A_1^2+A_2^2)(A_1^2+A_2^2+B_2^2)} \Big] + \frac{2}{\pi\sqrt{\pi}} \Omega_2 \delta_2^2 C_p \theta_1^2 \frac{\partial \mu_1}{\partial x} A_2 \left[\frac{1}{\sqrt{2}\, A_2^3} \tan^{-1}\left(\frac{B_2}{\sqrt{2}\, A_2}\right) + \frac{B_2}{A_2^2(2A_2^2+B_2^2)} \right] + \frac{4}{\pi\sqrt{\pi}} C_p \theta_1^2 \mu_1 \frac{\partial B_1}{\partial x} A_1 \frac{\Omega_1 \delta_1 \delta_2}{(A_1^2+A_2^2+B_1^2)^2}$$

$$+ \frac{4}{\pi\sqrt{\pi}} C_p \theta_1^2 \mu_1 \frac{\partial B_1}{\partial x} A_2 \frac{\Omega_1 \delta_2^2}{(2A_2^2+B_1^2)^2} + \frac{2}{\pi\sqrt{\pi}} C_p \theta_1^2 \frac{A_1}{B_1} \frac{\partial \mu_1}{\partial x} \frac{\Omega_1 \delta_1 \delta_2}{(A_1^2+A_2^2+B_1^2)} + \frac{2}{\pi\sqrt{\pi}} C_p \theta_1^2 \frac{A_2}{B_1} \frac{\partial \mu_1}{\partial x} \frac{\Omega_1 \delta_2^2}{(2A_2^2+B_1^2)}$$

$$- \frac{2}{\pi\sqrt{\pi}} C_p \theta_1^2 \mu_1 \frac{A_1}{B_1^2} \frac{\partial B_1}{\partial x} \frac{\Omega_1 \delta_1 \delta_2}{(A_1^2+A_2^2+B_1^2)} - \frac{2}{\pi\sqrt{\pi}} C_p \theta_1^2 \mu_1 \frac{A_2}{B_1^2} \frac{\partial B_1}{\partial x} \frac{\Omega_1 \delta_2^2}{(2A_2^2+B_1^2)} - \frac{4}{\pi\sqrt{\pi}} \Omega_1 \delta_1 \delta_2 C_p \theta_1^2 \mu_1 \frac{\partial B_1}{\partial x} \frac{A_1}{(A_1^2+A_2^2+B_1^2)^2}$$

$$- \frac{4}{\pi\sqrt{\pi}} \Omega_1 \delta_2^2 C_p \theta_1^2 \mu_1 \frac{\partial B_1}{\partial x} \frac{A_2}{(2A_2^2+B_1^2)^2} + \frac{4}{\pi\sqrt{\pi}} C_p \Omega_2 \delta_1 \delta_2 \theta_1^2 \mu_1 \frac{\partial B_2}{\partial x} \frac{A_1}{(A_1^2+A_2^2+B_2^2)^2} + \frac{4}{\pi\sqrt{\pi}} \Omega_2 \delta_2^2 C_p \theta_1^2 \mu_1 \frac{\partial B_2}{\partial x} \frac{A_2}{(2A_2^2+B_2^2)^2}$$

$$+ \frac{2}{\pi\sqrt{\pi}} C_p \theta_1^2 \frac{A_1}{B_2} \frac{\partial \mu_1}{\partial x} \frac{\Omega_2 \delta_1 \delta_2}{(A_1^2+A_2^2+B_2^2)} + \frac{2}{\pi\sqrt{\pi}} C_p \theta_1^2 \frac{\partial \mu_1}{\partial x} \frac{A_2}{B_2} \frac{\Omega_2 \delta_2^2}{(2A_2^2+B_2^2)} - \frac{2}{\pi\sqrt{\pi}} C_p \theta_1^2 \mu_1 \frac{\partial B_2}{\partial x} \frac{A_1}{B_2^2} \frac{\Omega_2 \delta_1 \delta_2}{(A_1^2+A_2^2+B_2^2)}$$

$$- \frac{2}{\pi\sqrt{\pi}} C_p \theta_1^2 \mu_1 \frac{\partial B_2}{\partial x} \frac{A_2}{B_2^2} \frac{\Omega_2 \delta_2^2}{(2A_2^2+B_2^2)} - \frac{4}{\pi\sqrt{\pi}} \Omega_2 \delta_1 \delta_2 C_p \theta_1^2 \mu_1 \frac{\partial B_2}{\partial x} \frac{A_1}{(A_1^2+A_2^2+B_2^2)^2} - \frac{4}{\pi\sqrt{\pi}} \Omega_2 \delta_2^2 C_p \theta_1^2 \mu_1 \frac{\partial B_2}{\partial x} \frac{A_2}{(2A_2^2+B_2^2)^2}$$

$$- \frac{2}{\pi\sqrt{\pi}} C_p \theta_1^2 \frac{\partial \mu_1}{\partial x} \frac{A_1}{B_1} \frac{\Omega_1 \delta_1 \delta_2}{(A_1^2+A_2^2)} - \frac{1}{\pi\sqrt{\pi}} C_p \theta_1^2 \frac{\partial \mu_1}{\partial x} \frac{\Omega_1 \delta_2^2}{B_1 A_2} + \frac{2}{\pi\sqrt{\pi}} C_p \theta_1^2 \mu_1 \frac{\partial B_1}{\partial x} \frac{A_1}{B_1^2} \frac{\Omega_1 \delta_1 \delta_2}{(A_1^2+A_2^2)} + \frac{1}{\pi\sqrt{\pi}} C_p \theta_1^2 \mu_1 \frac{\partial B_1}{\partial x} \frac{\Omega_1 \delta_2^2}{B_1^2 A_2}$$

$$- \frac{2}{\pi\sqrt{\pi}} C_p \theta_1^2 \frac{\partial \mu_1}{\partial x} \frac{A_1}{B_2} \frac{\Omega_2 \delta_1 \delta_2}{(A_1^2+A_2^2)} - \frac{1}{\pi\sqrt{\pi}} C_p \theta_1^2 \frac{\partial \mu_1}{\partial x} \frac{\Omega_2 \delta_2^2}{B_2 A_2} + \frac{2}{\pi\sqrt{\pi}} C_p \theta_1^2 \mu_1 \frac{\partial B_2}{\partial x} \frac{A_1}{B_2^2} \frac{\Omega_2 \delta_1 \delta_2}{(A_1^2+A_2^2)} + \frac{1}{\pi\sqrt{\pi}} C_p \theta_1^2 \mu_1 \frac{\partial B_2}{\partial x} \frac{\Omega_2 \delta_2^2}{B_2^2 A_2}$$

$$- \frac{2}{\sqrt{\pi}} C_p \frac{\theta_1^2 A_1^3 C T_1 \rho_1 \delta_1 \delta_2}{\rho_0^2 P_r (A_1^2+A_2^2)^{1/2}(A_1^2+A_2^2)} - \frac{1}{2}\sqrt{\frac{2}{\pi}}\, \delta_2^2 C_p \frac{\theta_1^2 C T_1 \rho_1}{\rho_0^2 P_r} + \frac{4\Omega_1^2 \delta_2}{\pi\sqrt{\pi}} \left(\frac{\rho_1}{\rho_0}\right)^2 \frac{\theta_1^2 \mu_1^2 B_1^2 C T_1}{(A_2^2+2B_1^2)} + \frac{8}{\pi\sqrt{\pi}} \left(\frac{\rho_1}{\rho_0}\right)^2 \frac{\theta_1 \mu_1^2 B_1 B_2 C T_1 \Omega_2 \delta_2 \Omega_1}{\rho_1 (A_2^2+B_1^2+B_2^2)}$$

$$+ \frac{4}{\pi\sqrt{\pi}} \left(\frac{\rho_1}{\rho_0}\right)^2 \frac{\theta_1 \mu_1^2 B_2^2 C T_1 \Omega_2^2 \delta_2}{\rho_1 (A_2^2+2B_2^2)} + \frac{1}{\sqrt{\pi}} C_p \theta_1^2 \frac{\partial T_1}{\partial t} \frac{R \delta_1 \delta_2}{R\theta_1 + \eta P + R T_w} \frac{A_1}{A_2^2 (A_1^2+A_2^2)^{1/2}} + \frac{1}{2}\sqrt{\frac{2}{\pi}}\, C_p \frac{\theta_1^2}{A_2^2} \frac{\partial T_1}{\partial t} \frac{R \delta_2^2}{R\theta_1 + \eta P + R T_w}$$

$$+ \frac{B_1}{A_2^2(2A_2^2+B_1^2)} \Big] - \frac{2}{\pi\sqrt{\pi}} \Omega_2 \delta_1 \delta_2 C_p \theta_1 \frac{\partial \theta_1}{\partial x} \mu_1 \Big[\frac{A_1}{A_2^2(A_1^2+A_2^2)^{1/2}} \tan^{-1}\Big(\frac{B_2}{(A_1^2+A_2^2)^{1/2}}\Big) + \frac{B_2}{A_2^2(A_2^2+B_2^2)^{1/2}}$$

$$- \frac{2}{\pi\sqrt{\pi}} \Omega_2 \delta_2^2 C_p \theta_1 \frac{\partial \theta_1}{\partial x} \mu_1 \Big[\frac{1}{\sqrt{2}\,A_2^3} \tan^{-1}\Big(\frac{B_2}{\sqrt{2}\,A_2}\Big) + \frac{B_2}{A_2^2(A_2^2+B_2^2)^{1/2}} \tan^{-1}\Big(\frac{A_2}{(A_2^2+B_2^2)^{1/2}}\Big) \Big] - \frac{2}{\pi\sqrt{}}$$

$$+ \frac{B_2}{(A_1^2+A_2^2)(A_1^2+A_2^2+B_2^2)} \Big] - \frac{2}{\pi\sqrt{\pi}} \Omega_2 \delta_2^2 C_p \theta_1 \mu_1 \frac{\partial A_2}{\partial x} \Big[\frac{1}{\sqrt{2}\,A_2^3} \tan^{-1}\Big(\frac{B_2}{\sqrt{2}\,A_2}\Big) + \frac{B_2}{A_2^2(2A_2^2+B_2^2)} \Big]$$

$$- \frac{1}{\sqrt{\pi}} \Omega_2 \delta_2 C_p \theta_1 \mu_1 \frac{\partial T_w}{\partial x} \frac{B_2}{A_2^2(A_2^2+B_2^2)^{1/2}} + \frac{2}{\pi\sqrt{\pi}} \Omega_1 \delta_1 \delta_2 C_p \theta_1^2 \frac{\partial \mu_1}{\partial x} A_1 \Big[\frac{1}{(A_1^2+A_2^2)^{3/2}} \tan^{-1}\Big(\frac{}{(A_1^2+}$$

$$+ \frac{2}{\pi\sqrt{\pi}} \Omega_1 \delta_2^2 C_p \theta_1^2 \frac{\partial \mu_1}{\partial x} A_2 \Big[\frac{1}{\sqrt{2}\,A_2^3} \tan^{-1}\Big(\frac{B_1}{\sqrt{2}\,A_2}\Big) + \frac{B_1}{A_2^2(2A_2^2+B_1^2)} \Big] + \frac{2}{\pi\sqrt{\pi}} \Omega_2 \delta_1 \delta_2 C_p \theta_1^2 \frac{\partial \mu_1}{\partial x}$$

$$+ \frac{B_2}{(A_1^2+A_2^2)(A_1^2+A_2^2+B_2^2)} \Big] + \frac{2}{\pi\sqrt{\pi}} \Omega_2 \delta_2^2 C_p \theta_1^2 \frac{\partial \mu_1}{\partial x} A_2 \Big[\frac{1}{\sqrt{2}\,A_2^3} \tan^{-1}\Big(\frac{B_2}{\sqrt{2}\,A_2}\Big) + \frac{B_2}{A_2^2(2A_2^2+B_2^2)} \Big] + \frac{4}{\pi}$$

$$+ \frac{4}{\pi\sqrt{\pi}} C_p \theta_1^2 \mu_1 \frac{\partial B_1}{\partial x} A_2 \frac{\Omega_1 \delta_2^2}{(2A_2^2+B_1^2)^2} + \frac{2}{\pi\sqrt{\pi}} C_p \theta_1^2 \frac{A_1}{B_1} \frac{\partial \mu_1}{\partial x} \frac{\Omega_1 \delta_1 \delta_2}{(A_1^2+A_2^2+B_1^2)} + \frac{2}{\pi\sqrt{\pi}} C_p \theta_1^2 \frac{A_2}{B_1} \frac{\partial}{\partial}$$

$$- \frac{2}{\pi\sqrt{\pi}} C_p \theta_1^2 \mu_1 \frac{A_1}{B_1^2} \frac{\partial B_1}{\partial x} \frac{\Omega_1 \delta_1 \delta_2}{(A_1^2+A_2^2+B_1^2)} - \frac{2}{\pi\sqrt{\pi}} C_p \theta_1^2 \mu_1 \frac{A_2}{B_1^2} \frac{\partial B_1}{\partial x} \frac{\Omega_1 \delta_2^2}{(2A_2^2+B_1^2)} - \frac{4}{\pi\sqrt{\pi}} \Omega_1 \delta_1 \delta_2 C_p \theta$$

$$- \frac{4}{\pi\sqrt{\pi}} \Omega_1 \delta_2^2 C_p \theta_1^2 \mu_1 \frac{\partial B_1}{\partial x} \frac{A_2}{(2A_2^2+B_1^2)^2} + \frac{4}{\pi\sqrt{\pi}} C_p \Omega_2 \delta_1 \delta_2 \theta_1^2 \mu_1 \frac{\partial B_2}{\partial x} \frac{A_1}{(A_1^2+A_2^2+B_2^2)^2} + \frac{4}{\pi\sqrt{\pi}} \Omega$$

$$+ \frac{2}{\pi\sqrt{\pi}} C_p \theta_1^2 \frac{A_1}{B_2} \frac{\partial \mu_1}{\partial x} \frac{\Omega_2 \delta_1 \delta_2}{(A_1^2+A_2^2+B_2^2)} + \frac{2}{\pi\sqrt{\pi}} C_p \theta_1^2 \frac{\partial \mu_1}{\partial x} \frac{A_2}{B_2} \frac{\Omega_2 \delta_2^2}{(2A_2^2+B_2^2)} - \frac{2}{\pi\sqrt{\pi}} C_p \theta_1^2 \mu_1 \frac{\partial B_2}{\partial x} \frac{A_1}{B_2^2}$$

$$- \frac{2}{\pi\sqrt{\pi}} C_p \theta_1^2 \mu_1 \frac{\partial B_2}{\partial x} \frac{A_2}{B_2^2} \frac{\Omega_2 \delta_2^2}{(2A_2^2+B_2^2)} - \frac{4}{\pi\sqrt{\pi}} \Omega_2 \delta_1 \delta_2 C_p \theta_1^2 \mu_1 \frac{\partial B_2}{\partial x} \frac{A_1}{(A_1^2+A_2^2+B_2^2)^2} - \frac{4}{\pi\sqrt{\pi}} \Omega_2 \delta_2$$

$$- \frac{2}{\pi\sqrt{\pi}} C_p \theta_1^2 \frac{\partial \mu_1}{\partial x} \frac{A_1}{B_1} \frac{\Omega_1 \delta_1 \delta_2}{(A_1^2+A_2^2)} - \frac{1}{\pi\sqrt{\pi}} C_p \theta_1^2 \frac{\partial \mu_1}{\partial x} \frac{\Omega_1 \delta_2^2}{B_1 A_2} + \frac{2}{\pi\sqrt{\pi}} C_p \theta_1^2 \mu_1 \frac{\partial B_1}{\partial x} \frac{A_1}{B_1^2} \frac{\Omega_1 \delta_1 \delta_2}{(A_1^2+A_2^2)} +$$

$$- \frac{2}{\pi\sqrt{\pi}} C_p \theta_1^2 \frac{\partial \mu_1}{\partial x} \frac{A_1}{B_2} \frac{\Omega_2 \delta_1 \delta_2}{(A_1^2+A_2^2)} - \frac{1}{\pi\sqrt{\pi}} C_p \theta_1^2 \frac{\partial \mu_1}{\partial x} \frac{\Omega_2 \delta_2^2}{B_2 A_2} + \frac{2}{\pi\sqrt{\pi}} C_p \theta_1^2 \mu_1 \frac{\partial B_2}{\partial x} \frac{A_1}{B_2^2} \frac{\Omega_2 \delta_1 \delta_2}{(A_1^2+A_2^2)} +$$

$$- \frac{2}{\sqrt{\pi}} C_p \frac{\theta_1^2 A_1^3 C T_1 \rho_1 \delta_1 \delta_2}{\rho_0^2 P_r (A_1^2+A_2^2)^{1/2}(A_1^2+A_2^2)} - \frac{1}{2}\sqrt{\frac{2}{\pi}}\, \delta_2^2 C_p \frac{\theta_1^2 C T_1 \rho_1}{\rho_0^2 P_r} + \frac{4\Omega_1^2 \delta_2}{\pi\sqrt{\pi}} \Big(\frac{\rho_1}{\rho_0}\Big)^2 \frac{\theta_1^2 \mu_1^2 B_1^2 C T_1}{(A_2^2+2B_1^2)} + \frac{8}{\pi}$$

$$+ \frac{4}{\pi\sqrt{\pi}} \Big(\frac{\rho_1}{\rho_0}\Big)^2 \frac{\theta_1 \mu_1^2 B_2^2 C T_1 \Omega_2^2 \delta_2}{\rho_1 (A_2^2+2B_2^2)} + \frac{1}{\sqrt{\pi}} C_p \theta_1^2 \frac{\partial T_1}{\partial t} \frac{R \delta_1 \delta_2}{R\theta_1 + \eta P + R T_w} \frac{A_1}{A_2^2(A_1^2+A_2^2)^{1/2}} + \frac{1}{2}\sqrt{\frac{2}{\pi}}$$

$$\frac{A_1}{A_2^2(A_1^2+A_2^2)^{1/2}} \tan^{-1}\left(\frac{B_2}{(A_1^2+A_2^2)^{1/2}}\right) + \frac{B_2}{A_2^2(A_2^2+B_2^2)^{1/2}} \tan^{-1}\left(\frac{A_1}{(A_2^2+B_2^2)^{1/2}}\right)\Big]$$

$$\left.\frac{\quad}{A_2}\right) + \frac{B_2}{A_2^2(A_2^2+B_2^2)^{1/2}} \tan^{-1}\left(\frac{A_2}{(A_2^2+B_2^2)^{1/2}}\right)\Big] - \frac{2}{\pi\sqrt{\pi}}\Omega_2\delta_1\delta_2 C_p\theta_1^2\mu_1\frac{\partial A_1}{\partial x}\Big[\frac{1}{(A_1^2+A_2^2)^{3/2}}\tan^{-1}\left(\frac{B_2}{(A_1^2+A_2^2)^{1/2}}\right)$$

$$\mu_1\frac{\partial A_2}{\partial x}\Big[\frac{1}{\sqrt{2}A_2^3}\tan^{-1}\left(\frac{B_2}{\sqrt{2}A_2}\right) + \frac{B_2}{A_2^2(2A_2^2+B_2^2)}\Big] - \frac{1}{\sqrt{\pi}}\Omega_1\delta_2 C_p\theta_1\mu_1\frac{\partial T_w}{\partial x}\frac{B_1}{A_2^2(A_2^2+B_1^2)^{1/2}}$$

$$\Omega_1\delta_1\delta_2 C_p\theta_1^2\frac{\partial\mu_1}{\partial x}A_1\Big[\frac{1}{(A_1^2+A_2^2)^{3/2}}\tan^{-1}\left(\frac{B_1}{(A_1^2+A_2^2)^{1/2}}\right) + \frac{B_1}{(A_1^2+A_2^2)(A_1^2+A_2^2+B_1^2)}\Big]$$

$$\Big) + \frac{B_1}{A_2^2(2A_2^2+B_1^2)}\Big] + \frac{2}{\pi\sqrt{\pi}}\Omega_2\delta_1\delta_2 C_p\theta_1^2\frac{\partial\mu_1}{\partial x}A_1\Big[\frac{1}{(A_1^2+A_2^2)^{3/2}}\tan^{-1}\left(\frac{B_2}{(A_1^2+A_2^2)^{1/2}}\right)$$

$$A_2\Big[\frac{1}{\sqrt{2}A_2^3}\tan^{-1}\left(\frac{B_2}{\sqrt{2}A_2}\right) + \frac{B_2}{A_2^2(2A_2^2+B_2^2)}\Big] + \frac{4}{\pi\sqrt{\pi}}C_p\theta_1^2\mu_1\frac{\partial B_1}{\partial x}A_1\frac{\Omega_1\delta_1\delta_2}{(A_1^2+A_2^2+B_1^2)^2}$$

$$C_p\theta_1^2\frac{A_1}{B_1}\frac{\partial\mu_1}{\partial x}\frac{\Omega_1\delta_1\delta_2}{(A_1^2+A_2^2+B_1^2)} + \frac{2}{\pi\sqrt{\pi}}C_p\theta_1^2\frac{A_2}{B_1}\frac{\partial\mu_1}{\partial x}\frac{\Omega_1\delta_2^2}{(2A_2^2+B_1^2)}$$

$$C_p\theta_1^2\mu_1\frac{A_2}{B_1^2}\frac{\partial B_1}{\partial x}\frac{\Omega_1\delta_2^2}{(2A_2^2+B_1^2)} - \frac{4}{\pi\sqrt{\pi}}\Omega_1\delta_1\delta_2 C_p\theta_1^2\mu_1\frac{\partial B_1}{\partial x}\frac{A_1}{(A_1^2+A_2^2+B_1^2)^2}$$

$$C_p\Omega_2\delta_1\delta_2\theta_1^2\mu_1\frac{\partial B_2}{\partial x}\frac{A_1}{(A_1^2+A_2^2+B_2^2)^2} + \frac{4}{\pi\sqrt{\pi}}\Omega_2\delta_2^2 C_p\theta_1^2\mu_1\frac{\partial B_2}{\partial x}\frac{A_2}{(2A_2^2+B_2^2)^2}$$

$$C_p\theta_1^2\frac{\partial\mu_1}{\partial x}\frac{A_2}{B_2}\frac{\Omega_2\delta_2^2}{(2A_2^2+B_2^2)} - \frac{2}{\pi\sqrt{\pi}}C_p\theta_1^2\mu_1\frac{\partial B_2}{\partial x}\frac{A_1}{B_2^2}\frac{\Omega_2\delta_1\delta_2}{(A_1^2+A_2^2+B_2^2)}$$

$$\Omega_2\delta_1\delta_2 C_p\theta_1^2\mu_1\frac{\partial B_2}{\partial x}\frac{A_1}{(A_1^2+A_2^2+B_2^2)^2} - \frac{4}{\pi\sqrt{\pi}}\Omega_2\delta_2^2 C_p\theta_1^2\mu_1\frac{\partial B_2}{\partial x}\frac{A_2}{(2A_2^2+B_2^2)^2}$$

$$\frac{\partial\mu_1}{\partial x}\frac{\Omega_1\delta_2^2}{B_1A_2} + \frac{2}{\pi\sqrt{\pi}}C_p\theta_1^2\mu_1\frac{\partial B_1}{\partial x}\frac{A_1}{B_1^2}\frac{\Omega_1\delta_1\delta_2}{(A_1^2+A_2^2)} + \frac{1}{\pi\sqrt{\pi}}C_p\theta_1^2\mu_1\frac{\partial B_1}{\partial x}\frac{\Omega_1\delta_2^2}{B_1^2A_2}$$

$$\frac{\partial\mu_1}{\partial x}\frac{\Omega_2\delta_2^2}{B_2A_2} + \frac{2}{\pi\sqrt{\pi}}C_p\theta_1^2\mu_1\frac{\partial B_2}{\partial x}\frac{A_1}{B_2^2}\frac{\Omega_2\delta_1\delta_2}{(A_1^2+A_2^2)} + \frac{1}{\pi\sqrt{\pi}}C_p\theta_1^2\mu_1\frac{\partial B_2}{\partial x}\frac{\Omega_2\delta_2^2}{B_2^2A_2}$$

$$\frac{\theta_1^2 C T_1\rho_1}{\rho_0^2 P_r} + \frac{4\Omega_1^2\delta_2}{\pi\sqrt{\pi}}\left(\frac{\rho_1}{\rho_0}\right)^2\frac{\theta_1^2\mu_1^2B_1^2CT_1}{(A_2^2+2B_1^2)} + \frac{8}{\pi\sqrt{\pi}}\left(\frac{\rho_1}{\rho_0}\right)^2\frac{\theta_1\mu_1^2B_1B_2CT_1\Omega_2\delta_2\Omega_1}{\rho_1(A_2^2+B_1^2+B_2^2)}$$

$$\frac{\partial T_1}{\partial t}\frac{R\delta_1\delta_2}{R\theta_1+\eta P+RT_w}\frac{A_1}{A_2^2(A_1^2+A_2^2)^{1/2}} + \frac{1}{2}\sqrt{\frac{2}{\pi}}C_p\frac{\theta_1^2}{A_2^2}\frac{\partial T_1}{\partial t}\frac{R\delta_2^2}{R\theta_1+\eta P+RT_w}$$

$$+\frac{1}{\sqrt{\pi}}\delta_2 C_p\frac{\partial T_1}{\partial t}\frac{\theta_1}{A_2^2}\frac{\eta P+RT_w}{R\theta_1+\eta P+RT_w}+\frac{1}{\sqrt{\pi}}C_p\theta_1^2\mu_1\frac{\partial T_1}{\partial x}\frac{R\delta_1\delta_2}{R\theta_1+\eta P+RT_w}\frac{A_1}{A_2^2(A_1^2+A_2^2)^{1/2}}+\frac{1}{2}\sqrt{\frac{2}{\pi}}C_p\frac{\theta_1^2}{A_2^2}\mu_1\frac{\partial T_1}{\partial x}\frac{R\delta_2^2}{R\theta_1+\eta P+RT_w}$$

$$+\frac{1}{\sqrt{\pi}}\delta_2 C_p\frac{\theta_1}{A_2^2}\mu_1\frac{\partial T_1}{\partial x}\frac{\eta P+RT_w}{R\theta_1+\eta P+RT_w}-\frac{2}{\pi\sqrt{\pi}}\theta_1^2\mu_1\frac{\partial \mu_1}{\partial t}\frac{R\Omega_1\delta_1\delta_2}{R\theta_1+\eta P+RT_w}\Big[\frac{A_1}{A_2^2(A_1^2+A_2^2)^{1/2}}\tan^{-1}\Big(\frac{B_1}{(A_1^2+A_2^2)^{1/2}}\Big)$$

$$+\frac{B_1}{A_2^2(A_2^2+B_1^2)^{1/2}}\tan^{-1}\Big(\frac{A_1}{(A_2^2+B_1^2)^{1/2}}\Big)\Big]-\frac{2}{\pi\sqrt{\pi}}\theta_1^2\mu_1\frac{\partial \mu_1}{\partial t}\frac{R\Omega_1\delta_2^2}{R\theta_1+\eta P+RT_w}\Big[\frac{1}{\sqrt{2}A_2^2}\tan^{-1}\Big(\frac{B_1}{\sqrt{2}A_2}\Big)+\frac{B_1}{A_2^2(A_2^2+B_1^2)^{1/2}}\tan^{-1}\Big(\frac{A_2}{(A_2^2+B_1^2)^{1/2}}\Big)\Big]$$

$$-\frac{1}{\sqrt{\pi}}\theta_1\mu_1\frac{\partial \mu_1}{\partial t}\frac{\eta P+RT_w}{R\theta_1+\eta P+RT_w}\frac{B_1\Omega_1\delta_2}{A_2^2(A_2^2+B_1^2)^{1/2}}-\frac{2}{\pi\sqrt{\pi}}\theta_1^2\mu_1\frac{\partial \mu_1}{\partial t}\frac{R\Omega_2\delta_1\delta_2}{R\theta_1+\eta P+RT_w}\Big[\frac{A_1}{A_2^2(A_1^2+A_2^2)^{1/2}}\tan^{-1}\Big(\frac{B_2}{(A_1^2+A_2^2)^{1/2}}\Big)$$

$$+\frac{B_2}{A_2^2(A_2^2+B_2^2)^{1/2}}\tan^{-1}\Big(\frac{A_1}{(A_2^2+B_2^2)^{1/2}}\Big)\Big]-\frac{2}{\pi\sqrt{\pi}}\theta_1^2\mu_1\frac{\partial \mu_1}{\partial t}\frac{R\Omega_2\delta_2^2}{R\theta_1+\eta P+RT_w}\Big[\frac{1}{\sqrt{2}A_2^2}\tan^{-1}\Big(\frac{B_2}{\sqrt{2}A_2}\Big)+\frac{B_2}{A_2^2(A_2^2+B_2^2)^{1/2}}\tan^{-1}\Big(\frac{A_2}{(A_2^2+B_2^2)^{1/2}}\Big)\Big]$$

$$-\frac{1}{\sqrt{\pi}}\theta_1\mu_1\frac{\partial \mu_1}{\partial t}\frac{\eta P+RT_w}{R\theta_1+\eta P+RT_w}\frac{B_2\Omega_2\delta_2}{A_2^2(A_2^2+B_2^2)^{1/2}}-\frac{2}{\pi\sqrt{\pi}}\theta_1^2\mu_1^2\frac{\partial \mu_1}{\partial x}\frac{R\Omega_1\delta_1\delta_2}{R\theta_1+\eta P+RT_w}\Big[\frac{A_1}{A_2^2(A_1^2+A_2^2)^{1/2}}\tan^{-1}\Big(\frac{B_1}{(A_1^2+A_2^2)^{1/2}}\Big)$$

$$+\frac{B_1}{A_2^2(A_2^2+B_1^2)^{1/2}}\tan^{-1}\Big(\frac{A_1}{(A_2^2+B_1^2)^{1/2}}\Big)\Big]-\frac{2}{\pi\sqrt{\pi}}\theta_1^2\mu_1^2\frac{\partial \mu_1}{\partial x}\frac{R\Omega_1\delta_2^2}{R\theta_1+\eta P+RT_w}\Big[\frac{1}{\sqrt{2}A_2^2}\tan^{-1}\Big(\frac{B_1}{\sqrt{2}A_2}\Big)+\frac{B_1}{A_2^2(A_2^2+B_1^2)^{1/2}}\tan^{-1}\Big(\frac{A_2}{(A_2^2+B_1^2)^{1/2}}\Big)\Big]$$

$$-\frac{1}{\sqrt{\pi}}\theta_1\mu_1^2\frac{\partial \mu_1}{\partial x}\frac{\eta P+RT_w}{R\theta_1+\eta P+RT_w}\frac{B_1\Omega_1\delta_2}{A_2^2(A_2^2+B_1^2)^{1/2}}-\frac{2}{\pi\sqrt{\pi}}\theta_1^2\mu_1^2\frac{\partial \mu_1}{\partial x}\frac{R\Omega_2\delta_1\delta_2}{R\theta_1+\eta P+RT_w}\Big[\frac{A_1}{A_2^2(A_1^2+A_2^2)^{1/2}}\tan^{-1}\Big(\frac{B_2}{(A_1^2+A_2^2)^{1/2}}\Big)$$

$$+\frac{B_2}{A_2^2(A_2^2+B_2^2)^{1/2}}\tan^{-1}\Big(\frac{A_1}{(A_2^2+B_2^2)^{1/2}}\Big)\Big]-\frac{2}{\sqrt{\pi}}\theta_1^2\mu_1^2\frac{\partial \mu_1}{\partial x}\frac{R\Omega_2\delta_2^2}{R\theta_1+\eta P+RT_w}\Big[\frac{1}{\sqrt{2}A_2^2}\tan^{-1}\Big(\frac{B_2}{\sqrt{2}A_2}\Big)+\frac{B_2}{A_2^2(A_2^2+B_2^2)^{1/2}}\tan^{-1}\Big(\frac{A_2}{(A_2^2+B_2^2)^{1/2}}\Big)\Big]$$

$$-\frac{1}{\sqrt{\pi}}\theta_1\mu_1^2\frac{\partial \mu_1}{\partial x}\frac{\eta P+RT_w}{R\theta_1+\eta P+RT_w}\frac{B_2\Omega_2\delta_2}{A_2^2(A_2^2+B_2^2)^{1/2}}+\frac{1}{\sqrt{\pi}}\theta_1^2\mu_1\frac{\partial \mu_1}{\partial t}\frac{R\delta_1\delta_2}{R\theta_1+\eta P+RT_w}\frac{A_1}{A_2^2(A_1^2+A_2^2)^{1/2}}+\frac{1}{2}\sqrt{\frac{2}{\pi}}\frac{\theta_1^2}{A_2^2}\mu_1\frac{\partial \mu_1}{\partial t}\frac{R\delta_2^2}{R\theta_1+\eta P+RT_w}$$

$$+\frac{1}{\sqrt{\pi}}\delta_2\frac{\theta_1}{A_2^2}\mu_1\frac{\partial \mu_1}{\partial t}\frac{\eta P+RT_w}{R\theta_1+\eta P+RT_w}+\frac{1}{\sqrt{\pi}}\theta_1^2\mu_1^2\frac{\partial \mu_1}{\partial x}\frac{R\delta_1\delta_2}{R\theta_1+\eta P+RT_w}\frac{A_1}{A_2^2(A_1^2+A_2^2)^{1/2}}+\frac{1}{2}\sqrt{\frac{2}{\pi}}\frac{\theta_1^2}{A_2^2}\mu_1^2\frac{\partial \mu_1}{\partial x}\frac{R\delta_2^2}{R\theta_1+\eta P+RT_w}$$

$$+\frac{1}{\sqrt{\pi}}\delta_2\theta_1\frac{\mu_1^2}{A_2^2}\frac{\partial \mu_1}{\partial x}\frac{\eta P+RT_w}{R\theta_1+\eta P+RT_w}\Big\}\Big/\frac{1}{4}\sqrt{\frac{2}{\pi}}\delta_2^2\frac{1}{A_2^2} \qquad [3.29]$$

Thereby, the objective stated in the beginning of this Section has been achieved. The resulting equations are Equations 3.24, 3.25, 3.28 and 3.29. These are still partial differential equations with only two independent variables. The independent variable x will be eliminated in the next [illegible].

$$+ \frac{1}{\sqrt{\pi}} \delta_2 C_p \frac{\partial T_1}{\partial t} \frac{\theta_1}{A_2^2} \frac{\eta P + RT_w}{R\theta_1 + \eta P + RT_w} + \frac{1}{\sqrt{\pi}} C_p \theta_1^2 \mu_1 \frac{\partial T_1}{\partial x} \frac{R\delta_1\delta_2}{R\theta_1 + \eta P + RT_w} \frac{A_1}{A_2^2 (A_1^2 + A_2^2)^{1/2}} + \frac{1}{2} \sqrt{\quad}$$

$$+ \frac{1}{\sqrt{\pi}} \delta_2 C_p \frac{\theta_1}{A_2^2} \mu_1 \frac{\partial T_1}{\partial x} \frac{\eta P + RT_w}{R\theta_1 + \eta P + RT_w} - \frac{2}{\pi\sqrt{\pi}} \theta_1^2 \mu_1 \frac{\partial \mu_1}{\partial t} \frac{R\Omega_1\delta_1\delta_2}{R\theta_1 + \eta P + RT_w} \Big[\frac{A_1}{A_2^2 (A_1^2 + A_2^2)^{1/2}} \tan^{-1} ($$

$$+ \frac{B_1}{A_2^2 (A_2^2 + B_1^2)^{1/2}} \tan^{-1} \left(\frac{A_1}{(A_2^2 + B_1^2)^{1/2}} \right) \Big] - \frac{2}{\pi\sqrt{\pi}} \theta_1^2 \mu_1 \frac{\partial \mu_1}{\partial t} \frac{R\Omega_1\delta_2^2}{R\theta_1 + \eta P + RT_w} \Big[\frac{1}{\sqrt{2} A_2^2} \tan^{-1} \left(\frac{B_1}{\sqrt{2} A_2} \right)$$

$$- \frac{1}{\sqrt{\pi}} \theta_1 \mu_1 \frac{\partial \mu_1}{\partial t} \frac{\eta P + RT_w}{R\theta_1 + \eta P + RT_w} \frac{B_1\Omega_1\delta_2}{A_2^2 (A_2^2 + B_1^2)^{1/2}} - \frac{2}{\pi\sqrt{\pi}} \theta_1^2 \mu_1 \frac{\partial \mu_1}{\partial t} \frac{R\Omega_2\delta_1\delta_2}{R\theta_1 + \eta P + RT_w} \Big[\frac{A}{A_2^2 (A_1^2}$$

$$+ \frac{B_2}{A_2^2 (A_2^2 + B_2^2)^{1/2}} \tan^{-1} \left(\frac{A_1}{(A_2^2 + B_2^2)^{1/2}} \right) \Big] - \frac{2}{\pi\sqrt{\pi}} \theta_1^2 \mu_1 \frac{\partial \mu_1}{\partial t} \frac{R\Omega_2\delta_2^2}{R\theta_1 + \eta P + RT_w} \Big[\frac{1}{\sqrt{2} A_2^2} \tan^{-1} (\frac{B_2}{\sqrt{2} A}$$

$$- \frac{1}{\sqrt{\pi}} \theta_1 \mu_1 \frac{\partial \mu_1}{\partial t} \frac{\eta P + RT_w}{R\theta_1 + \eta P + RT_w} \frac{B_2\Omega_2\delta_2}{A_2^2 (A_2^2 + B_2^2)^{1/2}} - \frac{2}{\pi\sqrt{\pi}} \theta_1^2 \mu_1^2 \frac{\partial \mu_1}{\partial x} \frac{R\Omega_1\delta_1\delta_2}{R\theta_1 + \eta P + RT_w} \Big[\frac{A_1}{A_2^2 (A_1^2 +}$$

$$+ \frac{B_1}{A_2^2 (A_2^2 + B_1^2)^{1/2}} \tan^{-1} \left(\frac{A_1}{(A_2^2 + B_1^2)^{1/2}} \right) \Big] - \frac{2}{\pi\sqrt{\pi}} \theta_1^2 \mu_1^2 \frac{\partial \mu_1}{\partial x} \frac{R\Omega_1\delta_2^2}{R\theta_1 + \eta P + RT_w} \Big[\frac{1}{\sqrt{2} A_2^2} \tan^{-1} (\frac{B_1}{\sqrt{2} A_2}$$

$$- \frac{1}{\sqrt{\pi}} \theta_1 \mu_1^2 \frac{\partial \mu_1}{\partial x} \frac{\eta P + RT_w}{R\theta_1 + \eta P + RT_w} \frac{B_1\Omega_1\delta_2}{A_2^2 (A_2^2 + B_1^2)^{1/2}} - \frac{2}{\pi\sqrt{\pi}} \theta_1^2 \mu_1^2 \frac{\partial \mu_1}{\partial x} \frac{R\Omega_2\delta_1\delta_2}{R\theta_1 + \eta P + RT_w} \Big[\frac{A_1}{A_2^2 (A_1^2 +}$$

$$+ \frac{B_2}{A_2^2 (A_2^2 + B_2^2)^{1/2}} \tan^{-1} \left(\frac{A_1}{(A_2^2 + B_2^2)^{1/2}} \right) \Big] - \frac{2}{\sqrt{\pi}} \theta_1^2 \mu_1^2 \frac{\partial \mu_1}{\partial x} \frac{R\Omega_2\delta_2^2}{R\theta_1 + \eta P + RT_w} \Big[\frac{1}{\sqrt{2} A_2^2} \tan^{-1} (\frac{B_2}{\sqrt{2} A_2}$$

$$- \frac{1}{\sqrt{\pi}} \theta_1 \mu_1^2 \frac{\partial \mu_1}{\partial x} \frac{\eta P + RT_w}{R\theta_1 + \eta P + RT_w} \frac{B_2\Omega_2\delta_2}{A_2^2 (A_2^2 + B_2^2)^{1/2}} + \frac{1}{\sqrt{\pi}} \theta_1^2 \mu_1 \frac{\partial \mu_1}{\partial t} \frac{R\delta_1\delta_2}{R\theta_1 + \eta P + RT_w} \frac{A_1}{A_2^2 (A_1^2 + A_2^2)}$$

$$+ \frac{1}{\sqrt{\pi}} \delta_2 \frac{\theta_1}{A_2^2} \mu_1 \frac{\partial \mu_1}{\partial t} \frac{\eta P + RT_w}{R\theta_1 + \eta P + RT_w} + \frac{1}{\sqrt{\pi}} \theta_1^2 \mu_1^2 \frac{\partial \mu_1}{\partial x} \frac{R\delta_1\delta_2}{R\theta_1 + \eta P + RT_w} \frac{A_1}{A_2^2 (A_1^2 + A_2^2)^{1/2}} + \frac{1}{2} \sqrt{\frac{2}{\pi}}$$

$$+ \frac{1}{\sqrt{\pi}} \delta_2 \theta_1 \frac{\mu_1^2}{A_2^2} \frac{\partial \mu_1}{\partial x} \frac{\eta P + RT_w}{R\theta_1 + \eta P + RT_w} \Big\} \Big/ \frac{1}{4} \sqrt{\frac{2}{\pi}} \delta_2^2 \frac{1}{A_2^3}$$

Thereby, the objective stated in the beginning of this Section has been achieved. Equations 3.24, 3.25, 3.28 and 3.29. These are still partial differential equation independent variables. The independent variable x will be eliminated in the next S

$$\frac{1}{\sqrt{\pi}} C_p \theta_1^2 \mu_1 \frac{\partial T_1}{\partial x} \frac{R\delta_1\delta_2}{R\theta_1+\eta P+RT_w} \frac{A_1}{A_2^2(A_1^2+A_2^2)^{1/2}} + \frac{1}{2}\sqrt{\frac{2}{\pi}}\, C_p \frac{\theta_1^2}{A_2^2}\mu_1 \frac{\partial T_1}{\partial x}\frac{R\delta_2^2}{R\theta_1+\eta P+RT_w}$$

$$- \frac{2}{\pi\sqrt{\pi}} \theta_1^2 \mu_1 \frac{\partial \mu_1}{\partial t} \frac{R\Omega_1\delta_1\delta_2}{R\theta_1+\eta P+RT_w}\left[\frac{A_1}{A_2^2(A_1^2+A_2^2)^{1/2}}\tan^{-1}\left(\frac{B_1}{(A_1^2+A_2^2)^{1/2}}\right)\right.$$

$$\Big] - \frac{2}{\pi\sqrt{\pi}} \theta_1^2 \mu_1 \frac{\partial \mu_1}{\partial t} \frac{R\Omega_1\delta_2^2}{R\theta_1+\eta P+RT_w}\left[\frac{1}{\sqrt{2}A_2^2}\tan^{-1}\left(\frac{B_1}{\sqrt{2}A_2}\right)+\frac{B_1}{A_2^2(A_2^2+B_1^2)^{1/2}}\tan^{-1}\left(\frac{A_2}{(A_2^2+B_1^2)^{1/2}}\right)\right]$$

$$\frac{\Omega_1\delta_2}{+B_1^2)^{1/2}} - \frac{2}{\pi\sqrt{\pi}} \theta_1^2 \mu_1 \frac{\partial \mu_1}{\partial t} \frac{R\Omega_2\delta_1\delta_2}{R\theta_1+\eta P+RT_w}\left[\frac{A_1}{A_2^2(A_1^2+A_2^2)^{1/2}}\tan^{-1}\left(\frac{B_2}{(A_1^2+A_2^2)^{1/2}}\right)\right.$$

$$\Big)\Big] - \frac{2}{\pi\sqrt{\pi}} \theta_1^2 \mu_1 \frac{\partial \mu_1}{\partial t} \frac{R\Omega_2\delta_2^2}{R\theta_1+\eta P+RT_w}\left[\frac{1}{\sqrt{2}A_2^2}\tan^{-1}\left(\frac{B_2}{\sqrt{2}A_2}\right)+\frac{B_2}{A_2^2(A_2^2+B_2^2)^{1/2}}\tan^{-1}\left(\frac{A_2}{(A_2^2+B_2^2)^{1/2}}\right)\right]$$

$$\frac{B_2\Omega_2\delta_2}{A_2^2+B_2^2)^{1/2}} - \frac{2}{\pi\sqrt{\pi}} \theta_1^2 \mu_1^2 \frac{\partial \mu_1}{\partial x} \frac{R\Omega_1\delta_1\delta_2}{R\theta_1+\eta P+RT_w}\left[\frac{A_1}{A_2^2(A_1^2+A_2^2)^{1/2}}\tan^{-1}\left(\frac{B_1}{(A_1^2+A_2^2)^{1/2}}\right)\right.$$

$$\Big] - \frac{2}{\pi\sqrt{\pi}} \theta_1^2 \mu_1^2 \frac{\partial \mu_1}{\partial x} \frac{R\Omega_1\delta_2^2}{R\theta_1+\eta P+RT_w}\left[\frac{1}{\sqrt{2}A_2^2}\tan^{-1}\left(\frac{B_1}{\sqrt{2}A_2}\right)+\frac{B_1}{A_2^2(A_2^2+B_1^2)^{1/2}}\tan^{-1}\left(\frac{A_2}{(A_2^2+B_1^2)^{1/2}}\right)\right]$$

$$\frac{\Omega_1\delta_2}{+B_1^2)^{1/2}} - \frac{2}{\pi\sqrt{\pi}} \theta_1^2 \mu_1^2 \frac{\partial \mu_1}{\partial x} \frac{R\Omega_2\delta_1\delta_2}{R\theta_1+\eta P+RT_w}\left[\frac{A_1}{A_2^2(A_1^2+A_2^2)^{1/2}}\tan^{-1}\left(\frac{B_2}{(A_1^2+A_2^2)^{1/2}}\right)\right.$$

$$\Big] - \frac{2}{\sqrt{\pi}} \theta_1^2 \mu_1^2 \frac{\partial \mu_1}{\partial x} \frac{R\Omega_2\delta_2^2}{R\theta_1+\eta P+RT_w}\left[\frac{1}{\sqrt{2}A_2^2}\tan^{-1}\left(\frac{B_2}{\sqrt{2}A_2}\right)+\frac{B_2}{A_2^2(A_2^2+B_2^2)^{1/2}}\tan^{-1}\left(\frac{A_2}{(A_2^2+B_2^2)^{1/2}}\right)\right]$$

$$\frac{\Omega_2\delta_2}{+B_2^2)^{1/2}} + \frac{1}{\sqrt{\pi}} \theta_1^2 \mu_1 \frac{\partial \mu_1}{\partial t} \frac{R\delta_1\delta_2}{R\theta_1+\eta P+RT_w}\frac{A_1}{A_2^2(A_1^2+A_2^2)^{1/2}} + \frac{1}{2}\sqrt{\frac{2}{\pi}}\frac{\theta_1^2}{A_2^2}\mu_1\frac{\partial \mu_1}{\partial t}\frac{R\delta_2^2}{R\theta_1+\eta P+RT_w}$$

$$+ \frac{1}{\sqrt{\pi}} \theta_1^2 \mu_1^2 \frac{\partial \mu_1}{\partial x} \frac{R\delta_1\delta_2}{R\theta_1+\eta P+RT_w}\frac{A_1}{A_2^2(A_1^2+A_2^2)^{1/2}} + \frac{1}{2}\sqrt{\frac{2}{\pi}}\frac{\theta_1^2}{A_2^2}\mu_1^2\frac{\partial \mu_1}{\partial x}\frac{R\delta_2^2}{R\theta_1+\eta P+RT_w}$$

$$\Big/ \frac{1}{4}\sqrt{\frac{2}{\pi}}\,\delta_2^2\,\frac{1}{A_2^3} \qquad [3.29]$$

the beginning of this Section has been achieved. The resulting equations are 29. These are still partial differential equations but with only two endent variable x will be eliminated in the next Section.

3.2 Method of Lines

The Method of Lines[39] is a technique commonly used to solve partial differential equations on general-purpose analog computers. When two independent variables exist, say x and t as in the present problem, one independent variable is discretized and the other is permitted to remain continuous, usually with respect to time t. The partial differential equation is thereby converted to a system of ordinary differential equations. General-purpose analog computers are intrinsically capable of handling only ordinary differential equations.

The usual technique of solving partial differential equations on a digital computer involves the approximating of derivatives with respect to all independent variables by finite difference expressions. The resulting algebraic relationships are then solved by matrix methods at the grid points of the discretized region of interest. Another possibility for the digital computer solution is to convert to a system of ordinary differential equations as in the Method of Lines and then to solve the system by standard techniques such as Runga-Kutta or Predictor-Corrector methods available for ordinary differential equations. This is the approach that will be used in the present investigation. The established numerical techniques with automatic step-size control and automatic error control for solving ordinary differential equations can be used. Moreover, Smith and Clutter[40] solved several boundary layer problems by the method in conjunction with high-speed digital computers. Koob and Abbott[41] solved the unsteady incompressible boundary layer equations without pressure gradients by use of the Method of Lines and the method of weighted residuals. Hicks and Wei[42] solved the unsteady one-dimensional heat diffusion equation by the Method of Lines. This approach yields results with less computer time than the explicit or implicit finite-difference methods.

The basic assumption of the Method of Lines is that continuous functions such as B_1, B_2, A_1, and A_2 and one of their derivatives such as

$\frac{\partial B_1}{\partial x}$, $\frac{\partial B_2}{\partial x}$, $\frac{\partial A_1}{\partial x}$ and $\frac{\partial A_2}{\partial x}$ can be approximated in terms of

several new functions such as $B_1(x_1)$, $B_1(x_2)$,$\cdots B_1(x_n)$, $B_2(x_1)$,$\cdots B_2(x_n)$, $A_1(x_1)$,$\cdots A_1(x_n)$, $A_2(x_1)$,$\cdots A_2(x_n)$,

$$\frac{B_1(x_2)-B_1(x_1)}{x_2-x_1},\ldots\frac{B_1(x_n)-B_1(x_{n-1})}{x_n-x_{n-1}},\ \frac{B_2(x_2)-B_2(x_1)}{x_2-x_1},$$

$$\ldots\frac{B_2(x_n)-B_2(x_{n-1})}{x_n-x_{n-1}},\ \frac{A_1(x_2)-A_1(x_1)}{x_2-x_1},\ldots\frac{A_1(x_n)-A_1(x_{n-1})}{x_n-x_{n-1}},$$

$$\ldots\frac{A_2(x_2)-A_2(x_1)}{x_2-x_1}\ldots\frac{A_2(x_n)-A_2(x_{n-1})}{x_n-x_{n-1}}\ .$$

Here, the field of interest is assumed to be between x_1 and x_n. The derivative of continuous functions is written in a backward difference form. If the information above is incorporated into Equations 3.23, 3.24, 3.28, and 3.29, a system of ordinary differential equations (4(n-1)) results with time as the only independent variable. The locations x_1 to x_n need not be at equal intervals. The solution of the resulting equations will be discussed in the next section.

3.3 Boundary Layer Parameters

The objective of the present investigation is to obtain the solution of unsteady compressible boundary layer equations stated in Section 2. The end results are in the form of velocity and temperature profiles as a function of the independent variables x and t. Once these profiles are determined, any other parameter of the boundary layer can be easily obtained. The derivation or evaluation of these parameters is given below:

Heat Transfer: The heat transfer at the wall by conduction is given by

$$Q_w = -\left(K\frac{\partial T}{\partial y}\right)_{y=0} = h(T_w-T_1) \tag{3.30}$$

Instead of introducing a model for thermal conductivity K which is a function of temperature, another form of Equation 3.30 may prove to be useful.

$$Q_w = -\left[\frac{\mu C_p}{Pr}\frac{\rho}{\rho_o}\frac{\partial\theta}{\partial\bar{y}}\right]_{\bar{y}=0} \tag{3.31}$$

The following final form is obtained by use of Equation 3.11.

$$Q_w = - \frac{2\mu_w C_{p,w} \rho_w \theta_1}{Pr \ \rho_o \sqrt{\pi}} (\delta_1 A_1 + \delta_2 A_2) \tag{3.32}$$

The convective heat transfer coefficient as defined by Equation 3.30 becomes

$$h = \frac{2\mu_w C_{p,w} \rho_w}{Pr \ \rho_o \ \sqrt{\pi}} (\delta_1 A_1 + \delta_2 A_2)$$

or $$Nu_w = \frac{2D}{\sqrt{\pi}} \frac{\rho_w}{\rho_o} (\delta_1 A_1 + \delta_2 A_2)$$

or $$St = - \frac{Q_w}{C_{p,w} \theta_1 \rho_1 u_1} = \frac{2\mu_w \rho_w}{\sqrt{\pi} \rho_o \rho_1 u_1 Pr} (\delta_1 A_1 + \delta_2 A_2) \tag{3.33}$$

Shear Stress: Assuming a Newtonian fluid, the shear stress at the wall in Cartesian coordinates is

$$\tau_w = \left(\mu \frac{\partial u}{\partial y}\right)_{y=0} \tag{3.34}$$

This can be written in terms of new variables as

$$\tau_w = \frac{2\mu_w \rho_w u_1}{\sqrt{\pi} \ \rho_o} (\Omega_1 B_1 + \Omega_2 B_2) \tag{3.35}$$

Finally, the dimensionless skin friction coefficient becomes

$$c_f = \frac{\tau_w}{\frac{1}{2} \rho_1 u_1^{\ 2}} = \frac{4\mu_w \rho_w}{\sqrt{\pi} \rho_1 \rho_o u_1} (\Omega_1 B_1 + \Omega_2 B_2) \tag{3.36}$$

Displacement thickness (δ_d): The boundary layer displacement thickness is defined as

$$\delta_d = \int_0^\infty (1 - \frac{\rho u}{\rho_1 u_1}) dy \tag{3.37}$$

Equation 3.37 becomes, in terms of variables in the present analysis,

$$\delta_d = \int_0^\infty [\rho_o(\frac{R\theta}{p} + \frac{RT_w}{p} + \eta) - \frac{\rho_o}{\rho_1}\frac{u}{u_1}] d\bar{y} \tag{3.38}$$

The following final form was obtained after substituting Equation 3.11 and integrating analytically:

$$\delta_d = \frac{\rho_o R\theta_1}{p}[\delta_1(\delta - \frac{1}{A_1\sqrt{\pi}}) + (\delta - \frac{1}{A_2\sqrt{\pi}})\delta_2] + \rho_o(\frac{RT_w}{p} + \eta)\delta$$

$$- \frac{\rho_o}{\rho_1}[\Omega_1(\delta - \frac{1}{B_1\sqrt{\pi}}) + \Omega_2 \; (\delta - \frac{1}{B_2\sqrt{\pi}})] \tag{3.39}$$

The displacement thickness indicates the distance by which the external streamlines are shifted outward owing to the formation of boundary layer.

Momentum thickness (δ_m):

$$\delta_m = \int_0^\infty \frac{\rho u}{\rho_1 u_1}(1 - \frac{u}{u_1}) \, dy$$

$$= \frac{\rho_o}{\rho_1} \int_0^\infty [\frac{u}{u_1} - (\frac{u}{u_1})^2] \, d\bar{y} \tag{3.40}$$

This parameter is useful in the determination of laminar-turbulent transition and also indicates a measure of loss of momentum in the boundary layer.

Energy-Dissipation thickness (δ_{ed}):

$$\delta_{ed} = \int_0^\infty \frac{\rho}{\rho_1} \frac{u}{u_1} (1 - \frac{u}{u_1})^2 \, dy$$

$$= \frac{\rho_o}{\rho_1} \int_0^\infty [\frac{u}{u_1} - (\frac{u}{u_1})^3] \, d\bar{y} \qquad (3.41)$$

The parameter above indicates a loss of mechanical energy occurring in the boundary layer.

Enthalpy thickness (δ_h):

$$\delta_h = \int_0^\infty \frac{\rho}{\rho_1} \frac{u}{u_1} (\frac{T}{T_1} - 1) \, dy$$

$$= \frac{\rho_o}{\rho_1} \int_0^\infty \frac{u}{u_1} [\frac{\theta}{T_1} + (\frac{T_w}{T_1} - 1)] \, d\bar{y} \qquad (3.42)$$

Velocity thickness (δ_u):

$$\delta_u = \int_0^\infty (1 - \frac{u}{u_1}) \, dy \qquad (3.43)$$

This integral cannot be evaluated unless y is written in terms of $\bar{y}$ by Equation 3.1. That is,

$$y = \rho_o \int_0^{\bar{y}} \left(\frac{R\theta}{p} + \frac{R}{p} T_w + \eta\right) d\bar{y}$$

$$= \frac{R\rho_o\theta_1}{p}[\delta_1(\bar{y} \text{ erf } (A_1\bar{y}) + \frac{e^{-A_1^2\bar{y}^2} - 1}{A_1 \sqrt{\pi}})$$

$$+ \delta_2(\bar{y} \text{ erf } (A_2\bar{y}) + \frac{e^{-A_2^2\bar{y}^2} - 1}{A_2 \sqrt{\pi}})] + \rho_o \left(\frac{RT_w}{p} + \eta\right) \bar{y} \qquad (3.44)$$

Even though the integrals in Equations 3.40, 3.41, 3.42, and 3.43 can be evaluated analytically, this is not attempted here because of the tedious derivations involved. However, one can easily evaluate these integrals by Simpson's rule of integration.

4. NUMERICAL ANALYSIS

The unsteady compressible boundary layer problem stated in Section 2 contains a system of nonlinear parabolic partial differential equations with three dependent variables (u, v, and T) and three independent variables (x, y and t). The introduction of the stream function ψ eliminated one of the dependent variables, namely v from the unknown list. The pressure and the free-stream velocity are to be provided either from experiments or from unsteady core flow analysis to close the system of equations. The variable, density, can be expressed in terms of pressure and temperature by means of an equation of state. The viscosity is considered as a function of temperature.

The elimination of the dependent variable v introduced a new dependent variable ψ to replace the original dependent variables u and v. The temperature T is replaced by a new dependent variable θ. The dependent variables ψ and θ are still a function of three independent variables x, $\bar{y}$, and t. The method of weighted residuals, in particular, the Galerkin method, introduced the dependent variables B_1 and B_2 to replace ψ and A_1 and A_2 to replace θ.

These dependent variables contain only the independent variables x and t (Equations 3.23, 3.24, 3.28 and 3.29. The Method of Lines made possible the transformation of these partial differential equations into ordinary differential equations with time as the only independent variable. However, this transformation is achieved only at the expense of introducing additional unknown dependent variables, a set of B_1's, B_2's, A_1's, and A_2's. This set is dependent upon the number of discretized x stations. In a typical example considered in Section 5, eleven x stations are chosen. Since a boundary initial-value problem has been transformed to an initial-value problem and thus requires specification of B_1, B_2, A_1, and A_2 at the first station as a function of time, 40 ordinary differential equations [4 times (11-1)] must be solved for that typical example to obtain the engineering accuracy. The number of equations will increase further if one increases either discretized x stations or the number of terms in the assumed approximate solution forms (Equation 3.11). The application of Method of Lines as explained in Section 3.2 on Equations 3.23, 3.24, 3.28, and 3.29 yields the following equations:

$$\dot{B}_1(x_2) = f_{1,2}(B_1(x_2),B_2(x_2),A_1(x_2),A_2(x_2))$$

$$\dot{B}_2(x_2) = f_{2,2}(B_1(x_2),B_2(x_2),A_1(x_2),A_2(x_2))$$

$$\dot{A}_1(x_2) = g_{1,2}(B_1(x_2),B_2(x_2),A_1(x_2),A_2(x_2))$$

$$\dot{A}_2(x_2) = g_{2,2}(B_1(x_2),B_2(x_2),A_1(x_2),A_2(x_2))$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\dot{B}_1(x_n) = f_{1,n}(B_1(x_n),B_2(x_n),A_1(x_n),A_2(x_n))$$

$$\dot{B}_2(x_n) = f_{2,n}(B_1(x_n),B_2(x_n),A_1(x_n),A_2(x_n))$$

$$\dot{A}_1(x_n) = g_{1,n}(B_1(x_n),B_2(x_n),A_1(x_n),A_2(x_n))$$

$$\dot{A}_2(x_n) = g_{2,n}(B_1(x_n),B_2(x_n),A_1(x_n),A_2(x_n)) \qquad (4.1)$$

Where the dot denotes differentiation with respect to time, t.

The functions f's and g's are the same as the right sides of Equations 3.23, 3.24, 3.28, and 3.29 with the exception of discretization of B's and A's and their derivatives.

The system of equations cited above may be solved either by standard Runga-Kutta techniques or by Predictor-Corrector methods. Since the functions in Equations 4.1 or in Equations 3.23, 3.24, 3.28, and 3.29 are complex and involve many tedious computations, some computer time may be saved by use of Predictor-Corrector methods. However, these predictor-corrector methods are not self-starting. Thus, initialization by other techniques such as the Runga-Kutta methods is desirable. Since the available time is limited and since, in the author's experience, the Runga-Kutta methods are more stable and more easily workable, the standard fourth order Runga-Kutta scheme is used for the entire investigation.

A fifth order Runga-Kutta integration scheme was also considered to improve the accuracy. The functions must be computed six times instead of four times in a fourth-order scheme for each time step. The additional computations introduce more errors. Moreover, Milne[43] states that the pursuit of greater accuracy by derivation of formulae of higher order is a "losing game." The formulae rapidly become formidable complexity, and the large number of substitutions for each time step increases computer time significantly for a slight increase in accuracy. Therefore, the fifth order Runga-Kutta method is not used ultimately.

Since the growth of the boundary layer is faster near the leading edge and slower downstream, the introduction of a variable step size for Δx is sometimes convenient. This can be accomplished easily without variable step sizes in numerical computations by means of the following transformation:

$$\text{Let} \qquad \chi = \sqrt{x/L} \tag{4.2}$$

where L is the length on the plate up to the end of the flow field under investigation. Since the variable x/L varies from 0 to 1, the equal step size of 0.1 can be obtained for x/L of 0, 0.01, 0.04, 0.09, 0.16, 0.25, 0.36, 0.49, 0.64, 0.81, and 1.0 (11 Stations). Thus one can obtain eight stations for the first half of the flow field and three for the remaining portion. This is achieved without a variable step size in the actual computer program. A few numerical examples will be considered in Section 5.

5. SAMPLE PROBLEMS

Not much literature is available on unsteady boundary layers even though hundreds of investigators, throughout the world, are working on boundary layers. This is especially true for compressible flow with pressure gradients. No known example exists that can be used directly as a test case. Koob and Abbott[41] solved an unsteady incompressible boundary layer problem on a semi-infinite flat plate with zero pressure gradients. A similar problem was also considered by Hall.[44] This analysis is based on implicit finite-difference methods. This example and the results will be discussed in Section 5.1.

Numerous studies were conducted by Mirels[45,46,47,48,49] for shock tube flow. Similarities are present between a shock tube flow and a gun tube flow. Indeed, these are identical with the limiting case of bullet mass approaching zero. The growth of the boundary layer starts at the high-pressure end as well as at the base of the shock. Similarity profiles were obtained by Mirels[45] for steady boundary layers. If the coordinate system is fixed to a shock wave, the flow in the boundary layer behind a shock wave may be interpreted as if the flow is steady. However, if the coordinate system is fixed to the shock tube, the same boundary layer behind a shock wave may be interpreted as an unsteady compressible boundary layer flow problem.[50] This is the only test known to the authors for an unsteady compressible boundary layer problem. This problem is discussed further in Section 5.2.

5.1 Rayleigh-Blasius Flow on a Flat Plate

No exact solution for unsteady compressible boundary layers apparently exists that would provide a complete test of the present method. The compressibility, large pressure gradients, viscous dissipation, and the gradients in time as large as the gradients in space display quite significant differences in results from steady state in an ideal test case. Since no such solution is known, the Rayleigh-Blasius incompressible flow on a flat plate may reveal at least some unsteady boundary layer characteristics.

The present method may be tested by computation of a solution that should, at initial times, be identical with Rayleigh's solution for an infinite flat plate started impulsively from rest and that should approach ultimately Blasius solution for a semi-infinite plate in a steady uniform stream. The boundary conditions on an upstream station are required for all times. However, these are unavailable for an impulsively started semi-infinite flat plate.

Therefore, arbitrary conditions (identical with Halls[44]) will be imposed for the purpose of this investigation. The Rayleigh and Blasius problems are discussed below to complete the presentation of the example problem.

The governing equations of the Rayleigh problem, or more commonly called "Stokes first problem" are

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \tag{5.1}$$

$t \leq 0$: $u=0$ for all y

$t>0$: $u=u_1$ for $y \simeq \infty$; $u=0$ for $y=0$

$$\text{Let} \quad \eta = \frac{y}{2\sqrt{\nu t}} \quad \text{and} \quad \frac{u}{u_1} = f(\eta) \tag{5.2}$$

If the definitions in Equation 5.2 are substituted into Equation 5.1, the following ordinary differential equation and corresponding boundary conditions are obtained:

$$f'' + 2\eta f' = 0$$

$$f(\infty) = 1, \; f(o) = 0 \tag{5.3}$$

where prime denotes differentiation with respect to η.

The solution of Equation 5.3 is

$$\frac{u}{u_1} = f(\eta) = \text{erf}\ (\eta) \tag{5.4}$$

Various parameters of boundary layers can be obtained easily by use of their definitions.

The governing equations of Blasius problem are

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$y=0: \quad u=0,\ v=0$$

$$y \doteq \infty: \quad u \quad u_1 \tag{5.5}$$

Let $\eta = \dfrac{y}{\sqrt{\dfrac{\nu x}{u_1}}}$ and $\psi = \sqrt{\nu x u_1}\ g(\eta)$ (5.6)

The following definitions eliminate the second equation of Equation 5.5 from further analysis.

$$u = \frac{\partial \psi}{\partial y} = u_1 g'(\eta)$$

$$v = -\frac{\partial \psi}{\partial x} = \tfrac{1}{2}\sqrt{\frac{\nu u_1}{x}}\ (\eta g' - g) \tag{5.7}$$

If the definitions from Equations 5.6 and 5.7 are substituted into Equation 5.5, the following equations are obtained:

$$2g''' + gg' = 0$$

$$g(o)=0,\ g'(o)=0,\ g'(\infty) = 1 \tag{5.8}$$

where prime denotes differentiation with respect to η.

No closed form solution exists except in series. The results by numerical integration are as follows:

TABLE I

$\eta = \dfrac{y}{\sqrt{\dfrac{\nu x}{u_1}}}$	g	$g' = \dfrac{u}{u_1}$	g''
0	0	0	0.33206
0.2	0.00664	0.06641	0.33199
0.4	0.02656	0.13277	0.33147
0.6	0.05974	0.19894	0.33008
0.8	0.10611	0.26471	0.32739
1.0	0.16551	0.32979	0.32301
1.2	0.23795	0.39378	0.31659
1.4	0.32298	0.45627	0.30787
1.6	0.42032	0.51676	0.29667
1.8	0.52952	0.57477	0.28293
2.0	0.65003	0.62977	0.26675
2.2	0.78120	0.68132	0.24835
2.4	0.92230	0.72899	0.22809
2.6	1.07252	0.77246	0.20646
2.8	1.23099	0.81152	0.18401
3.0	1.39682	0.84605	0.16136
3.2	1.56911	0.87609	0.13913
3.4	1.74696	0.90177	0.11788
3.6	1.92954	0.92333	0.09809
3.8	2.11605	0.94112	0.08013
4.0	2.30576	0.95552	0.06424
4.2	2.49806	0.96696	0.05052
4.4	2.69238	0.97587	0.03897
4.6	2.88826	0.98269	0.02948
4.8	3.08534	0.98779	0.02187
5.0	3.28329	0.99155	0.01591
5.2	3.48189	0.99425	0.0[illegible]134
5.4	3.68094	0.99616	0.00793
5.6	3.88031	0.99748	0.00543
5.8	4.07990	0.99838	0.00365
6.0	4.27926	0.99992	0.00022
7.2	5.47925	0.99996	0.00013
7.4	5.67924	0.99998	0.00007
7.6	5.87924	0.99999	0.00004
7.8	6.07923	1.00000	0.00002
8.0	6.27923	1.00000	0.00001
8.2	6.47923	1.00000	0.00001
8.4	6.67923	1.00000	0.00000

Since the solutions to the Rayleigh and Blasius problems are obtained, the test case can now be set up. The flow field characteristics, and the initial and upstream conditions are set for compression of the present method with Hall's implicit finite-difference results. The region of interest is $1 \leq x \leq 2$. The free stream velocity (u_1) is unity. The initial time is 0.5. The kinematic viscosity in the present analysis will be set to unity to match with Hall's nondimensional problem. The boundary layer development form the time t=0.5 is sought. The initial and boundary conditions are as follows:

$$t=0.5,\ 1\leq x\leq 2:\ u = \operatorname{erf}\left(\frac{y}{2\sqrt{\nu t}}\right)$$

$$x=1,\ 0.5\leq x\leq 1:\ u = \operatorname{erf}\left(\frac{y}{2\sqrt{\nu t}}\right)$$

$$x=1,\ 1.0\leq x<\infty:\ u=\operatorname{erf}\left(\frac{y}{2\sqrt{\nu t}}\right)+\left[\operatorname{erf}\left(\frac{y}{2\sqrt{\nu t}}\right)-u_1 g'(\eta)\right]\cdot e^{-.25(t-1)^2} \qquad (5.9)$$

The function $g'(\eta)$ is already listed above. The last upstream condition represents an exponential variation from the Rayleigh to Blasius flow. This is an arbitrary condition and not necessarily true for an impulsively started semi-infinite flat plate.

The initial and upstream conditions given above are not readily acceptable in the present computer program. These initial and upstream conditions must be interpreted in terms of B_1's and B_2's. The definition of u in the present analysis is given in Equation 3.11. The B_1 and B_2 may be obtained by the matching (collocation method) of this equation with Equation 5.9. Since these equations are of transcendental type, one must solve these equations for B_1 and B_2 by the multidimensional Newton-Raphson or another similar technique. Since this approach consumes more computer time and also since the conditions at the wall are more important than in the flow field for heat transfer or skin friction computations, the following alternative is considered. If the slope of the velocity component at the wall ($\partial u/\partial y$ at y=0) is matched, one of the unknowns can be expressed explicitly in terms of the other unknown. That is, B_1 in terms of B_2 or conversely:

$$\frac{\partial u}{\partial \bar{y}} = u_1\Omega_1\frac{2B_1}{\sqrt{\pi}}e^{-B_1^2\bar{y}^2} + u_1\Omega_2\frac{2B_2}{\sqrt{\pi}}e^{-B_2^2\bar{y}^2} \qquad (5.10)$$

$$\left.\frac{\partial u}{\partial \bar{y}}\right|_{\bar{y}=0} = \frac{2u_1}{\sqrt{\pi}}(\Omega_1 B_1+\Omega_2 B_2)$$

or $$B_1 = \left(\frac{\sqrt{\pi}}{2u_1}\left.\frac{\partial u}{\partial \bar{y}}\right|_{\bar{y}=0} - \Omega_2 B_2\right)/\Omega_1 \qquad (5.11)$$

Note that $\bar{y}$ is the same as y due to the incompressible assumption.

The resulting unknown may be found by the matching of Equation 5.10 with another point in the flow field near the wall. This is accomplished by the one-dimensional Newton-Raphson technique. The velocity gradient in Equations 5.10 and 5.11 can be obtained from Equation 5.9. The results are discussed in Section 6.

5.2 Shock-Induced Boundary Layer Problem

This is a compressible boundary layer problem but without pressure gradients. The governing equations of steady compressible boundary layers without pressure gradients were analyzed by Mirels[45] (similar to the analysis of the Blasius problem in Section 5.1) who integrated the resulting ordinary differential equations by numerical methods. The results were tabulated with dimensionless wall to the free-stream velocity as a parameter. The pertinent equations and a sample table ($u_W/u_1 = 2$) to interpret in terms of the variables in the present investigation are as follows:

$$\frac{u_W}{u_1} = \frac{(\gamma+1)\frac{u_W^2}{\gamma g R T_b}}{(\gamma\ 1)\frac{u_W^2}{\gamma g R T_b} + 2}$$

$$\frac{T_1}{T_b} = \frac{(\gamma+1)\frac{u_w}{u_1} - (\gamma-1)}{\frac{u_w}{u_1}(\gamma+1)-(\gamma-1)\frac{u_w}{u_1}}$$

$$\frac{T}{T_1} = 1+\left(\frac{u_w}{u_1} -1\right)^2 \frac{u_1^2 r(\eta)}{2T_1 C_{p,w}} + \left(\frac{T_w}{T_1} - \frac{T_r}{T_1}\right) s(\eta)$$

$$\frac{T_r}{T} = 1+\left(\frac{u_w}{u} -1\right)^2 \frac{u_1^2 r(o)}{2T_1 C_{p,w}}$$

$$\frac{u}{u_1} = f'(\eta)$$

$$\eta = \sqrt{\frac{u_1}{2x\nu_w}} \int_0^y \frac{T_w}{T} dy \qquad (5.12)$$

Note that the shock velocity and the wall velocity are the same to interpret the results between steady (coordinate system fixed to the shock wave) and unsteady (coordinate system fixed to the wall) flows. The shock speed can be obtained by the relationship mentioned above. For the unsteady case, the coordinates of flow of interest may be computed from the initial shock wave location, the shock speed, and the elapsed time. Thus, initial and upstream conditions may be obtained for the example under consideration.

TABLE II

Solution for $\frac{u_w}{u_1} = 2.0$,				Prandtl number $\sigma_w = 0.72$		
η	f'	f''	r	-r'	s	-s'
0.0	2.0000	1.0191	0.8997	0.0000	1.0000	0.8512
.1	1.8984	1.0091	.8922	.1479	.9151	.8452
.2	1.7988	.9804	.8704	.2861	.8313	.8279
.3	1.7029	.9356	.8356	.4068	.7498	.8004
.4	1.6121	.8777	.7898	.5043	.6715	.7645
.5	1.5276	.8103	.7356	.5758	.5972	.7217
.6	1.4503	.7367	.6755	.6209	.5274	.6739
.7	1.3304	.6601	.6122	.6412	.4625	.6227
.8	1.3183	.5834	.5480	.6398	.4029	.5697
.9	1.2637	.5088	.4849	.6206	.3486	.5162
1.0	1.2164	.4382	.4244	.5878	.2996	.4636
1.1	1.1759	.3728	.3676	.5455	.2558	.4127
1.2	1.1416	.3135	.3155	.4974	.2170	.3643
1.3	1.1129	.2606	.2683	.4465	.1829	.3189
1.4	1.0893	.2142	.2262	.3953	.1531	.2769
1.5	1.0699	.1742	.1892	.3456	.1274	.2386
1.6	1.0542	.1402	.1570	.2988	.1053	.2040
1.7	1.0417	.1116	.1293	.2556	.0864	.1731
1.8	1.0317	.0879	.1057	.2166	.0705	.1458
1.9	1.0239	.0686	.0858	.1819	.0572	.1219
2.0	1.0179	.0529	.0692	.1514	.0460	.1012
2.1	1.0133	.0404	.0554	.1250	.0368	.0833
2.2	1.0097	.0306	.0441	.1024	.0293	.0682
2.3	1.0071	.0229	.0348	.0832	.0231	.0553
2.4	1.0051	.0170	.0273	.0671	.0181	.0446
2.5	1.0036	.0124	.0213	.0538	.0141	.0357
2.6	1.0026	.0090	.0165	.0427	.0109	.0284
2.7	1.0018	.0065	.0127	.0337	.0084	.0224
2.8	1.0012	.0046	.0097	.0264	.0064	.0175
2.9	1.0008	.0033	.0073	.0205	.0049	.0136
3.0	1.0006	.0023	.0055	.0158	.0037	.0105
3.1	1.0004	.0016	.0041	.0121	.0028	.0081
3.2	1.0003	.0011	.0031	.0092	.0020	.0061
3.3	1.0002	.0007	.0023	.0070	.0015	.0046
3.4	1.0001	.0005	.0017	.0052	.0011	.0035
3.5	1.0001	.0003	.0012	.0039	.0008	.0026
3.6	1.0000	.0002	.0009	.0029	.0006	.0019
3.7	1.0000	.0001	.0006	.0021	.0004	.0014
3.8	1.0000	.0001	.0005	.0015	.0003	.0010
3.9	1.0000	.0001	.0003	.0011	.0002	.0007
4.0	1.0000	.0000	.0002	.0008	.0001	.0005
4.1			.0002	.0006	.0001	.0004
4.2			.0001	.0004	.0001	.0003
4.3			.0001	.0003	.0001	.0002
4.4			.0001	.0002	.0000	.0001
4.5			.0000	.0001	.0000	.0001
4.6			.0000	.0001	.0000	.0001
4.7			.0000	.0001	.0000	.0000
4.8			.0000	.0000	.0000	.0000
4.9						

Similar to the determination of B_1 and B_2 in the Rayleigh-Blasius flow problem in Section 5.1, not only B_1 and B_2 but also A_1 and A_2 are to be determined to provide initial and upstream conditions for the shock-induced boundary layer problem. The entire procedure is the same as before and need not be repeated here. The results are discussed in Section 6.

6. RESULTS AND CONCLUSIONS

The overall gun tube heat transfer problem applicable to any weapon, ammunition, and firing schedule was analyzed. Toward this goal, the propellant gas convective heat transfer problem was divided into five problems: (1) generation of thermochemical properties for any given propellant, (2) transient inviscid compressible flow through the gun barrel (core flow), (3) transient viscous compressible flow on the bore surface (boundary layers), (4) unsteady heat diffusion through single or multilayer gun tube, and (5) unsteady free convection and radiation outside the gun tube. Limited literature and solutions to these problems were discussed in Sections 1 and 2.

With NASA-LEWIS (CEC-71) thermochemical program (Chemical Equilibrium Chemistry), the chemical composition and adiabatic flame temperature and thermodynamic properties were computed for M18 and IMR. The unsteady inviscid core flow problem was solved by the method of characteristics. The transient two-dimensional heat diffusion through the gun tube wall was analyzed by finite-element methods. The unsteady, two-dimensional, free convection and radiation around gun tubes with variable wall temperature was analyzed by explicit finite-difference methods.

The transient, viscous, compressible flow on the bore surface with viscous dissipation and pressure gradients was formulated in Section 2. The unsteady compressible boundary layer on the bore surface is one of the most difficult problems to analyze due to the limited State of the Art (almost none) and also to the existence of laminar, transitional and turbulent regions within the boundary layer. Since the development of the transitional region is not well understood, the boundary layer is assumed to change suddenly from laminar to turbulent flow when the Reynolds' number, based on momentum thickness, reaches approximately 350. Similar assumption will be used for highly accelerated flow if laminarization occurs.

The governing equations of the unsteady compressible boundary layers are a system of nonlinear, parabolic, partial differential equations with three independent variables. The transverse coordinate was modified to absorb the compressibility effect. The Stream function was introduced to satisfy the continuity equation and also to eliminate one of the dependent variables. The method of weighted residuals was used to reduce by one the number of independent variables ($\bar{y}$). The approximate solution form was chosen based upon the asymptotic solution of the steady differential equations for large values of the spacelike coordinate. The error functions, consequently, occur in the solution form for forced convective boundary layer problems. The method of Galerkin was used as the error distribution principle. All integrations across the boundary layer were performed analytically. The Method of Lines was used to reduce the resulting partial differential equations in two independent variables to an approximate set of ordinary differential equations. This procedure enables one to solve for derivatives by the reduction of a matrix with elements of not more than the number of undetermined parameters introduced into the solution form, and thus less computer time is required. With this analysis, density and temperature variations are allowed not only across the boundary layer but also with time; also strong variations of free-stream parameters are allowed with both axial location and time. The various boundary layer parameters were derived in Section 3.3.

The resulting equations were programmed for a digital computer in Fortran IV. The standard fourth order Runga-Kutta method was used for numerical integration of a system of ordinary differential equations. The computer program with two terms (Equation 3.11) is listed in Appendix B. The typical output contains not only the profiles of velocity components (u and v), temperature and density but also the displacement thickness, momentum thickness, energy dissipation thickness, enthalpy thickness, velocity thickness, skin friction coefficient and convective heat transfer coefficient as a function of the Prandtl number. The Nobel-Abel equation of state was used to account for the imperfections caused by high-pressure powder gases.

First, an attempt was made to set up the computer logic and also to obtain the typical unsteady boundary layer characteristics with minimum labor. This objective can be achieved by preparation of a computer program with only the first term of Equation 3.11. Moreover, reasonable results

were anticipated with just one term. The results discussed in this section were obtained by this one-term computer program. The assumptions involved should be justified before interpretation of the results. The variation of parameter B_1 for the sample problem in Section 5.1 is shown in Figure 1. This figure indicates that the parameter B_1 is not a strong function of x, but that it varies significantly with time, t. Therefore, discretizing B_1 with respect to x, approximating linearly between the nodes (i.e., utilization of Method of Lines), and allowing B_1's to vary continuously with time, t, are justified. Also, B_1 decreases with increasing x to accommodate the growth of the boundary layer.

Next, an attempt was made to determine the convergence of the finite-difference scheme used in the Method of Lines by increasing the number of nodes from 9 to 17. This is the same as halving the spatial step size, Δx in the computations. This, in turn, doubles the number of ordinary differential equations to be solved either by standard Runga-Kutta or by other predictor-corrector methods. With the same time step (0.25) as in Figure 1 but with halved spatial step size (.0625), the results were unstable. However, good results were obtained by halving the time step whenever the spatial step was halved. Since B_1's and A_1's influence any physical parameter of the boundary layer (and the sample problem in Section 5.1 happened to be an uncompressible one), only B_1's are listed in Table III.

These results show that B_1's change only in the third or fourth digit after decimal even though both time and spatial step sizes were halved. If only spatial step size is halved by doubling the number of longitudinal stations, the changes in the value of B_1's can be interpreted as negligible.

The longitudinal velocity component is plotted in Figure 2. The present results are shown by solid lines. Hall's results are shown by dashed lines. The profiles at the upstream station (x=1.0) coincided with Hall's profiles for times 1.0 and 1.5 even though only one term of Equation 3.11 was utilized to obtain the present results. Slightly different results are obtained at upstream and downstream stations (x=2.0) for larger time (t>7). The difference is believed to be due to the use of different time-dependent boundary conditions (i.e., matching only the slope at the wall instead of the whole profile due to the use of only one term of Equation 3.11) instead of Hall's actual conditions. Since the results are quite satisfactory, even one term of Equation 3.11 can be concluded to yield reasonable results.

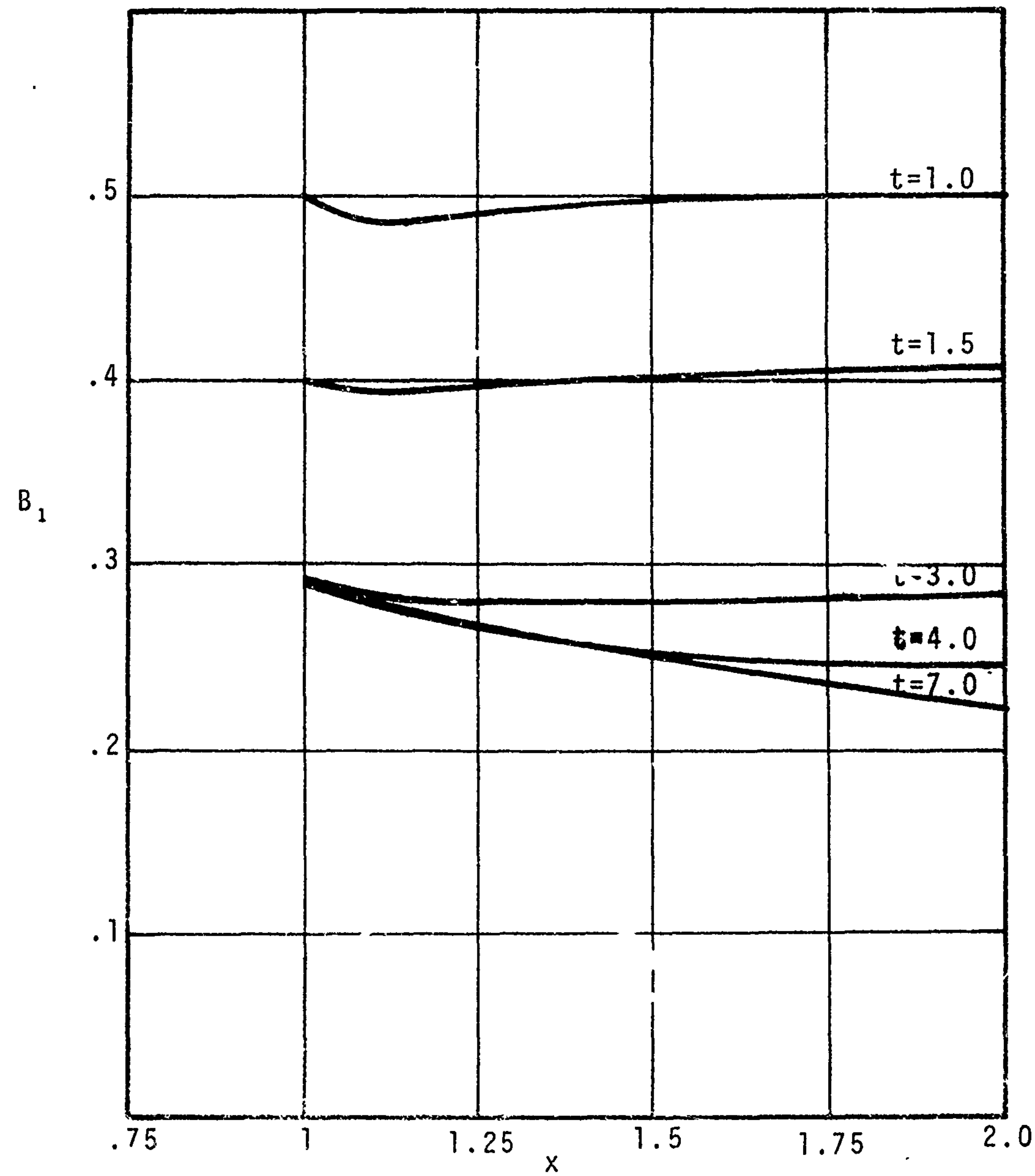

FIGURE 1 Streamwise Distribution of Parameter, B_1

TABLE III

x / Time	Δx = 0.1250				Δx = 0.0625			
	1.25	1.50	1.75	2.00	1.25	1.50	1.75	2.00
0.75	.5709	.5773	.5773	.5773	.5764	.5774	.5774	.5774
1.00	.4916	.4988	.4999	.5000	.4960	.4998	.5000	.5000
1.25	.4383	.4443	.4468	.4472	.4421	.4463	.4472	.4472
1.5	.3988	.4040	.4071	.4081	.4029	.4063	.4080	.4082
1.75	.3675	.3727	.3760	.3775	.3717	.3753	.3774	.3779
2.00	.3418	.3473	.3507	.3526	.3456	.3502	.3524	.3534
2.25	.3206	.3261	.3297	.3319	.3236	.3291	.3317	.3329
2.50	.3036	.3080	.3118	.3142	.3056	.3108	.3140	.3155
2.75	.2904	.2928	.2963	.2989	.2914	.2950	.2986	.3004
3.00	.2807	.2801	.2828	.2854	.2808	.2814	.2849	.2871
3.25	.2740	.2699	.2712	.2735	.2734	.2703	.2727	.2752

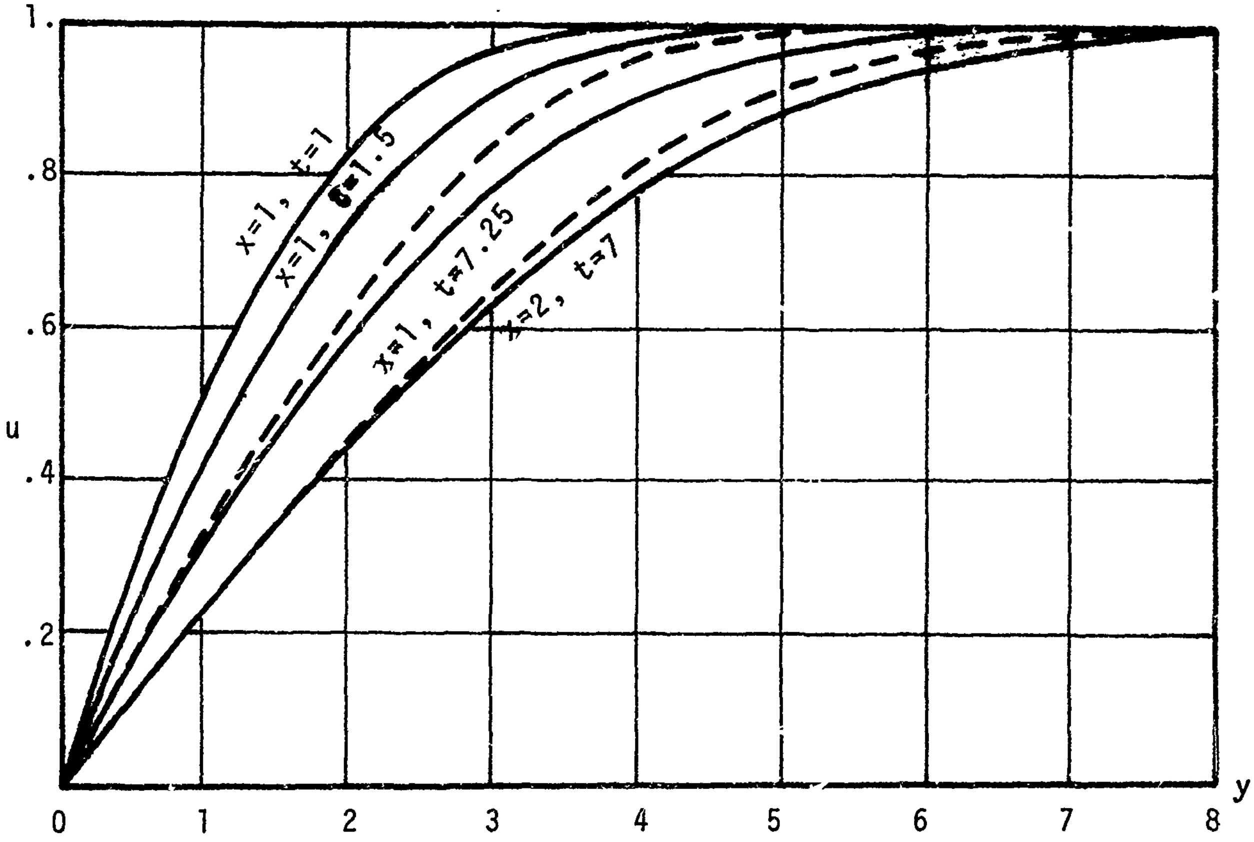

FIGURE 2 Profiles of Longitudinal Velocity Component

The transverse velocity component is shown in Figure 3. This is unavailable in Reference 44. However, the profile for larger times (t>7) is in agreement with the steady-state Blasius profile.[53]

The streamwise distribution of skin friction coefficient at various times is shown in Figure 4. At time t=1, the skin friction coefficient is uniform over most of the plate. Very little difference exists between the times 4 and 10 up to x=1.5. Finally, this is in good agreement with the steady-state Blasius skin friction coefficients. The transient displacement thickness is shown in Figure 5. The displacement thickness increases not only with time, t, but also with longitudinal coordinate, x. The present results are different from Blasius results (dashed) by approximately 5 per cent.

The two-term (Equation 3.11) computer program listed in Appendix B is not yet operational. This is yet to be "debugged" for reliable results. The numerical results of the sample problem stated in Section 5.2 were not obtained because of the expectation of similar results as above if only one term is used. The convergence of the terms in Equation 3.11 is yet to be proved, at best numerically, when the two-term computer program is operational. Reasonable results are somewhat surprising, with even one term of Equation 3.11.

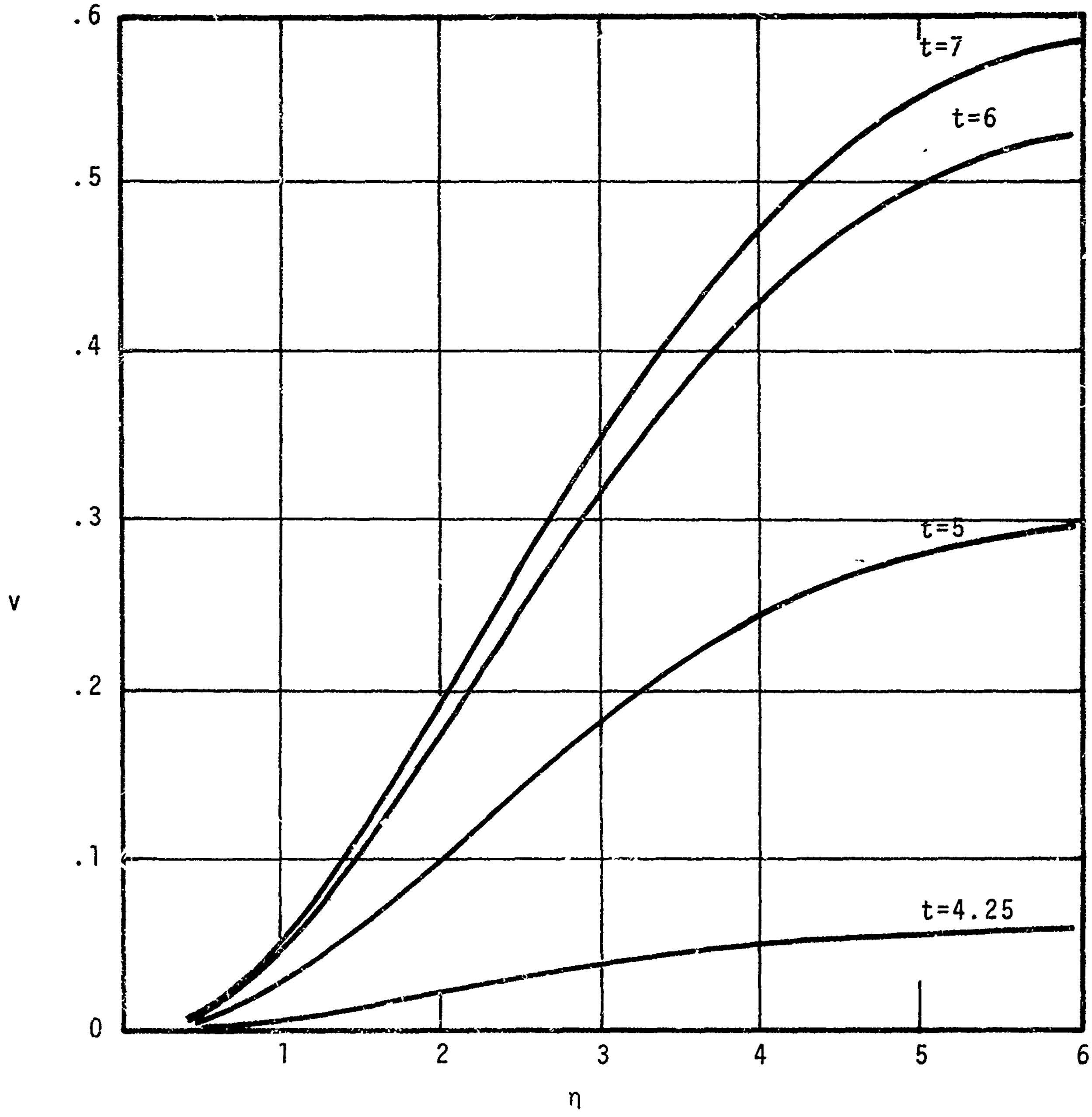

FIGURE 3 Profiles of Transverse Velocity Component

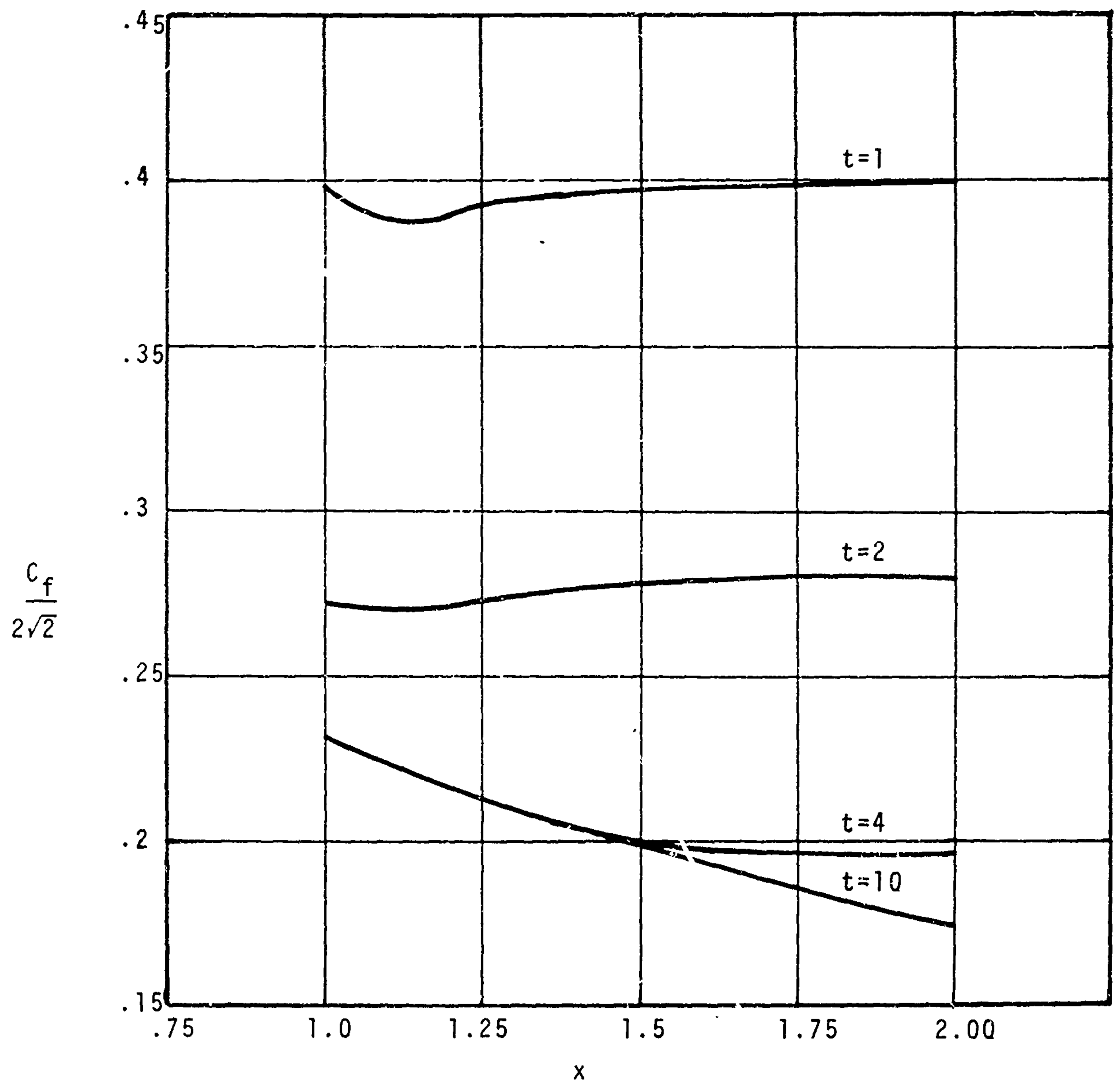

FIGURE 4 Distribution of Transient Skin Friction Coefficient

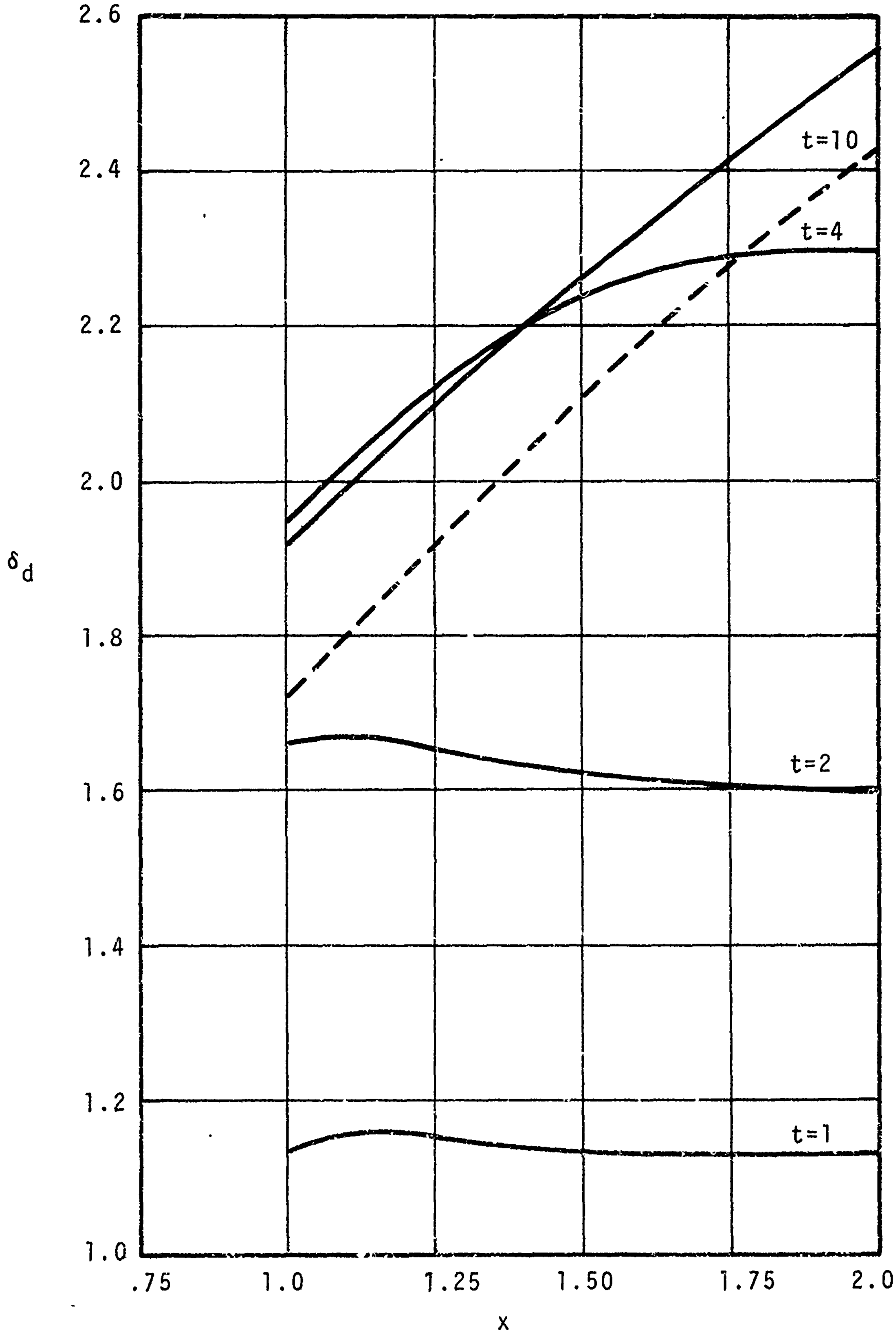

FIGURE 5 Distribution of Transient Displacement Thickness

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APPENDIX A

Evaluation of Integrals

APPENDIX A

$$\int_0^\infty \bar{\eta}\, \mathrm{erf}(B\bar{\eta})\, e^{-A^2\bar{\eta}^2}\, d\bar{\eta} \;;$$

let $u = \mathrm{erf}(B\bar{\eta})$ let $dv = \bar{\eta}\, e^{-A^2\bar{\eta}^2}\, d\bar{\eta}$

$du = \frac{2}{\sqrt{\pi}} B e^{-B^2\bar{\eta}^2}\, d\bar{\eta}$ $v = -\frac{1}{2A^2} e^{-A^2\bar{\eta}^2}$

$$\int u\, dv = uv - \int v\, du$$

$$\int_0^\infty \bar{\eta}\, \mathrm{erf}(B\bar{\eta})\, e^{-A^2\bar{\eta}^2}\, d\bar{\eta} = -\frac{1}{2A^2} \mathrm{erf}(B\bar{\eta})\, e^{-A^2\bar{\eta}^2} \Big|_0^\infty$$

$$+ \frac{1}{2A^2} \frac{2}{\sqrt{\pi}} B \int_0^\infty e^{-2B^2\bar{\eta}^2}\, d\bar{\eta}$$

$$\int_0^\infty \bar{\eta}\, \mathrm{erf}(B\bar{\eta})\, e^{-A^2\bar{\eta}^2}\, d\bar{\eta} = \frac{\sqrt{2}}{4A^2} \tag{A.1}$$

$$\int_0^\infty \bar{\eta}^2 e^{-A\bar{\eta}^2} \mathrm{erf}(B\bar{\eta})\, d\bar{\eta} = \left[\frac{1}{2\sqrt{\pi}\, A^{3/2}}\right] \tan^{-1}\left(\frac{B}{\sqrt{A}}\right) + \frac{B}{2\sqrt{\pi}\, A(B^2+A)} \qquad (A.2)^{54}$$

$$\int_0^\infty \bar{\eta}^2 \mathrm{erf}^2(B\bar{\eta})\, e^{-A\bar{\eta}^2}\, d\bar{\eta} = \frac{1}{\sqrt{\pi}\, A^{3/2}} \tan^{-1}\left(\sqrt{\frac{2B^2+A}{A}}\right) + \frac{B^2}{\sqrt{\pi}\, A(A+B^2)\sqrt{2B^2+A}} \qquad (A.3)^{54}$$

$$\int_0^\infty \eta \, \mathrm{erf}(A\eta)\, \mathrm{erf}(B\eta)\, e^{-C\eta^2} d\eta = \frac{A}{C\pi\sqrt{C+A^2}} \tan^{-1}\left(\frac{B}{\sqrt{C+A^2}}\right) + \frac{B}{C\pi\sqrt{C+B^2}} \tan^{-1}\left(\frac{A}{\sqrt{C+B^2}}\right) \quad (A.4)^{54}$$

$$\int_0^\infty X^2 \, \mathrm{erf}(Ax)\, \mathrm{erf}(Bx)\, e^{-Cx^2} dx = I$$

$$\mathrm{erf}(Ax) = Ax - \frac{A^3x^3}{3} + \frac{A^5x^5}{10} - \frac{A^7x^7}{42} + \frac{A^9x^9}{216} - \cdots$$

$$I = \int_0^\infty \left(Ax^3 - \frac{A^3}{3}x^5 + \frac{A^5}{10}x^7 - \frac{A^7}{42}x^9 + \cdots \right) e^{-Cx^2} \mathrm{erf}(Bx)\, dx$$

Taking each term of the series separately, one obtains integrals $A\int_0^\infty x^3 e^{-Cx^2} \mathrm{erf}(Bx)\, dx$, $\frac{A^3}{3}\int_0^\infty x^5 e^{-Cx^2} \mathrm{erf}(Bx)\, dx$, $\frac{A^5}{10}\int_0^\infty x^3 e^{-Cx^2} \mathrm{erf}(Bx)\, dx$, $\frac{A^7}{42}\int_0^\infty x^7 e^{-Cx^2} \mathrm{erf}(Bx)\, dx$, $\frac{A^9}{216}\int_0^\infty x^9 e^{-Cx^2} \mathrm{erf}(Bx)\, dx$, etc.

Consider $\int_0^\infty x^3 e^{-Cx^2} \mathrm{erf}(Bx)\, dx$. Let $dv = x^2 e^{-Cx^2} d\left(\frac{x^2}{2}\right)$, then $v = \frac{1}{2}\int_0^{x^2} p e^{-Cp} dp = -\frac{1}{2}\frac{e^{-Cx^2}}{A^2}(Cx^2+1)$.

Let $u = \mathrm{erf}(Bx)$, then $du = \frac{2}{\sqrt{\pi}} B e^{-B^2x^2} dx$.

$$\int u\, dv = uv - \int v\, du$$

$$\int_0^\infty x^3 e^{-Cx^2} \mathrm{erf}(Bx)\, dx = -\frac{1}{2}\frac{e^{-Cx^2}}{C^2}(Cx^2+1)\, \mathrm{erf}(Bx)\Big|_0^\infty + \frac{2B}{2C^2\sqrt{\pi}}\int_0^\infty (Cx^2+1)\, e^{-B^2x^2} e^{-Cx^2} dx$$

$$\int_0^\infty x^3 e^{-Cx^2} \mathrm{erf}(Bx)\, dx = 0 + \frac{B}{C^2\sqrt{\pi}}\left[\frac{C}{4(C+B^2)}\sqrt{\frac{\pi}{C+B^2}} + \frac{\sqrt{\pi}}{2\sqrt{C+B^2}}\right]$$

$$\therefore \int_0^\infty X^3 e^{-CX^2} erf(BX)\, dx = \frac{3CB + 2B^3}{4C^2(C+B^2)^{3/2}} \qquad (A.5)$$

Consider $\int_0^\infty X^5 e^{-CX^2} erf(BX)\, dx$.

Let $dv = X^4 e^{-CX^2} d\left(\frac{X^2}{2}\right)$, $\quad v = \frac{1}{2}\int_0^{X^2} p^2 e^{-Cp}\, dp$

$$v = -\left(\frac{X^4}{2C} + \frac{X^2}{C^2} + \frac{1}{C^3}\right) e^{-CX^2}$$

Let $u = erf(BX)$, $\quad du = \frac{2}{\sqrt{\pi}} B e^{-B^2X^2} dx$

$$\int_0^\infty X^5 e^{-CX^2} erf(BX)\, dx = -\left(\frac{X^4}{2C} + \frac{X^2}{C^2} + \frac{1}{C^3}\right) e^{-CX^2} erf(BX) \Big|_0^\infty$$

$$+ \frac{2B}{\sqrt{\pi}} \int_0^\infty \left(\frac{X^4}{2C} + \frac{X^2}{C^2} + \frac{1}{C^3}\right) e^{-(C+B^2)X^2} dX$$

$$\int_0^\infty X^5 e^{-CX^2} erf(BX)\, dx = \frac{2B}{\sqrt{\pi}} \left[\frac{1}{2C}\,\frac{3}{8(C+B^2)^2}\sqrt{\frac{\pi}{C+B^2}}\right.$$

$$\left. + \frac{1}{4C^2(C+B^2)}\sqrt{\frac{\pi}{C+B^2}} + \frac{\sqrt{\pi}}{2C^3\sqrt{C+B^2}}\right]$$

$$\therefore \int_0^\infty X^5 e^{-CX^2} erf(BX)\, dx = \frac{B}{(2C^3)(C+B^2)^{5/2}} \left[15C^2\right.$$

$$\left. + 20CB^2 + 8B^4\right] \qquad (A.6)$$

Consider $\int_0^\infty X^7 e^{-CX^2} erf(BX)\, dx$.

Let $dv = X^6 e^{-CX^2} d\left(\frac{X^2}{2}\right)$, $\quad v = \frac{1}{2}\int_0^{X^2} p^3 e^{-Cp}\, dp$

$$v = -\left(\frac{x^6}{2C} + \frac{3x^4}{2C^2} + \frac{3x^2}{C^3} + \frac{3}{C^4}\right) e^{-Cx^2}$$

Let $u = \operatorname{erf}(Bx)$, $du = \frac{2}{\sqrt{\pi}} B e^{-B^2 x^2} dx$

$$\int_0^\infty x^7 e^{-Cx^2} \operatorname{erf}(Bx)\, dx = -\left(\frac{x^6}{2C} + \frac{3x^4}{2C^2} + \frac{3x^2}{C^3} + \frac{3}{C^4}\right) e^{-Ax^2} \operatorname{erf}(Bx)\Big|_0^\infty$$

$$+ \frac{2}{\sqrt{\pi}} B \int_0^\infty \left(\frac{x^6}{2C} + \frac{3x^4}{2C^2} + \frac{3x^2}{C^3}\right.$$

$$\left. + \frac{3}{C^4}\right) e^{-(C+B^2)x^2}\, dx$$

$$\therefore \int_0^\infty x^7 e^{-Cx^2} \operatorname{erf}(Bx)\, dx = \frac{B}{(2C)^4 (C+B^2)^{7/2}} \left[105 C^3 + 210 C^2 B^2\right.$$

$$\left. + 168 C B^4 + 48 B^6\right] \qquad (A.7)$$

Utilizing the same technique as above,

one obtains $\int_0^\infty x^9 e^{-Cx^2} \operatorname{erf}(Bx)\, dx = \frac{B}{(2C)^5 (C+B^2)^{9/2}} \left[945 C^4\right.$

$\left. + 2520 C^3 B^2 + 3024 C^2 B^4 + 1728 C B^6 + 384 B^8\right]$ (A.8)

Combining these integrals, one gets

$$\int_0^\infty \bar{\eta}^2 \operatorname{erf}(A\bar{\eta}) \operatorname{erf}(B\bar{\eta}) e^{-C\bar{\eta}^2} d\bar{\eta} = \frac{AB(2B^2+C)}{4C^2(C+B^2)^{3/2}}$$

$$- \frac{A^3}{3} \frac{B(15C^2 + 20CB^2 + 8B^4)}{8C^3(C+B^2)^{5/2}} + \frac{A^5}{10} \frac{B(105C^3 + 210C^2B^2 + 168CB^4 + 48B^6)}{(2C)^4(C+B^2)^{7/2}}$$

$$- \frac{A^7}{42} \frac{B(945C^4 + 2520C^3B^2 + 3024C^2B^4 + 1728CB^6 + 384B^7)}{(2C)^5(C+B^2)^{9/2}}$$

$$+ \cdots \qquad (A.9)$$

APPENDIX B

Listing of Computer Program

```
//RAL JOB (--------,5),'P.BENZKOFER'
// EXEC WATFOR,TIME=2,REGION=28_K
//WAT.SYSIN DD *
$JOB 'P.BENZKOFER',TIME=12_,PAGES=160,KP=29
      IMPLICIT REAL*8(A-H),REAL*8(O-Z)
      EXTERNAL FU1,FU2,FU3,FU4
      COMMON OMEGA1,OMEGA2,R,PI,ETA,C,PR,DYB,P1,P2,P3,TW1,TW2
      COMMON TW3,T11,T12,T13,THT11,THT12,THT13,RHO0,RHO11,RHO12,RHO13
      COMMON DRHO11,DRHO12,DRHO13,VTWPX,VTHPX,DTW1,DTW2,DTW3,VL1,VL2
      COMMON VL3,DOMGA,VVP1,VVP2,VVP3,VVPP1,VVPP2,VVPP3,H,DT11,DT12,DT13
      COMMON C4,CC51,C42,CC2,CE5,CE3,CCE3,CCE5,CC5
      COMMON T1PX1,T1PX2,T1PX3,DELTA1,DELTA2,J1,CP
      COMMON DTHT11,DTHT12,DTHT13
      DIMENSION OMEGA(12),A1(12),A2(12),B1(12),B2(12),A1P(12),A2P(12)
      DIMENSION E1P(12),E2P(12),T1(12),T(12),VP(12),XL(12),THET1(12)
      DIMENSION PRE(12),VPP(12),RHO(12),DT(12),DTHT(12),DRHO(12)
      READ 1,N1,N2,N3,N4,N5,N6,N7,N8
      READ 2,(T1(I),I=1,N1)
      READ 2,(T(I),I=1,N2)
      READ 2,(VP(I),I=1,N3)
      READ 2,(XL(I),I=1,N4)
      READ 2,(THET1(I),I=1,N6)
      READ 2,(PRE(I),I=1,N7)
      READ 2,(VPP(I),I=1,N8)
      READ 2,(RHO(I),I=1,N8)
      READ 2,(DT(I),I=1,N8)
      READ 2,(DTHT(I),I=1,N8)
      READ 2,(DRHO(I),I=1,N8)
      READ 3,(OMEGA(I),I=1,11)
      PRINT 1,N1,N2,N3,N4,N5,N6,N7,N8
      PRINT 2,(T1(I),I=1,N1)
      PRINT 2,(T(I),I=1,N2)
      PRINT 2,(VP(I),I=1,N3)
      PRINT 2,(XL(I),I=1,N4)
      PRINT 2,(THET1(I),I=1,N6)
      PRINT 2,(PRE(I),I=1,N7)
      PRINT 2,(VPP(I),I=1,N8)
      PRINT 2,(RHO(I),I=1,N8)
      PRINT 2,(DT(I),I=1,N8)
      PRINT 2,(DTHT(I),I=1,N8)
      PRINT 2,(DRHO(I),I=1,N8)
      PRINT 3,(OMEGA(I),I=1,11)
    1 FORMAT(8I10)
    2 FORMAT(5F15.6)
    3 FORMAT(11F5.2)
      DOMGA=.125
      G=1
      R=1
      CP=1
      PI=3.14159
      OMEGA1=.7
```

```
      OMEGA2= .3
      DELTA1=.0
      DELTA2=.0
      ITEST=0
      US=1199.
      C1=2./DSQRT(PI)
      C3=1./C1
      C4=-1./DSQRT(PI)*OMEGA1*OMEGA2
      C49=OMEGA1**2*DSQRT(2./PI)/4.
      CC2=-OMEGA1*OMEGA2/DSQRT(PI)
      CC51=4.*OMEGA2**2/PI
      CE3=.25*CP*DSQRT(2./PI)
      CE5=-CP/DSQRT(PI)
      CCE3=-CP/DSQRT(PI)
      CCE5=.25*CP*DSQRT(2./PI)
      CC5=.25*DSQRT(2./PI)*OMEGA2**2
      TT=.5
      C=1
      ETA=0.
      TT1=10.
      PR=1
      RHOC=1
      J1=9
      DO 2  I=1,J1
      B1(I)=0.
      B2(I)=B1(I)
      A1(I)=B1(I)
      A2(I)=A1(I)
      A1P(I)=0.
      A2P(I)= .0
      B1P(I)= .0
      B2P(I)= .0
    2 CONTINUE
      TW1=1.
      TW2=1.
      TW3=1.
      VTWPX=0.0
      VTHPX=0.0
      DTW1=0.
      DTW2=0.0
      DTW3=0.0
      T1PX1=0.
      T1PX2=0.0
      T1PX3=0.0
      T11=1 60.
      T12=1.60.
      T13=1.60.
      VVP1=500.
      VVP2=500.
      VVP3=500.
      VL1=10.72
      VL2=10.72
      VL3=10.72
      THT11=T11-TW1
      THT12=T12-TW2
      THT13=T13-TW3
      P1=2117.
      P2=2117.
      P3=2117.
      VVPP1=0.0
      VVPP2=0.0
```

```
      VVPP3=0.
      RHO11=RHO.
      RHO12=RHO.
      RHO13=RHO.
      DT11=0.
      DT12=0.
      DT13=0.0
      DTHT11=0.
      DTHT12=0.
      DTHT13=0.
      DRHO1=0.0
      DRHO2=0.
      DRHO3=0.
      H=.125
      XT=1.
   10 J=1
      DYB=0.
      IF(ITEST.EQ.1) GO TO 11
      XS1=1.0
      GO TO 12
   11 XS1=US*H+XT
   12 XT=XS1
      IF(ITEST.EQ.0.) GO TO 523
      GO TO 323
  322 I=1
      CALL LINEAR(TT,T,T1,T11,I)
      CALL LINEAR(TT,T,VP,VVP1,I)
      CALL LINEAR(TT,T,XL,VL1,I)
      CALL LINEAR(TT,T,THET1,THT11,I)
      CALL LINEAR(TT,T,PRE,P1,I)
      CALL LINEAR(TT,T,VPP,VVPP1,I)
      CALL LINEAR(TT,T,RHO,RHO11,I)
      CALL LINEAR(TT,T,DT,DT11,I)
      CALL LINEAR(TT,T,DTHT,DTHT11,I)
      CALL LINEAR(TT,T,DRHO,DRHO1,I)
  323 TT=TT+H/2.
      XS2=US*H/2.+XT
      XT=XS2
      GO TO 423
  422 I=1
      CALL LINEAR(TT,T,T1,T12,I)
      CALL LINEAR(TT,T,VP,VVP2,I)
      CALL LINEAR(TT,T,XL,VL2,I)
      CALL LINEAR(TT,T,THET1,THT12,I)
      CALL LINEAR(TT,T,PRE,P2,I)
      CALL LINEAR(TT,T,VPP,VVPP2,I)
      CALL LINEAR(TT,T,RHO,RHO12,I)
      CALL LINEAR(TT,T,DT,DT12,I)
      CALL LINEAR(TT,T,DTHT,DTHT12,I)
      CALL LINEAR(TT,T,DRHO,DRHO2,I)
  423 TT=TT+H/2.
      XS3=US*H/2.+XT
      XT=XS3
      GO TO 523
  522 I=1
      CALL LINEAR(TT,T,T1,T13,I)
      CALL LINEAR(TT,T,VP,VVP3,I)
      CALL LINEAR(TT,T,XL,VL3,I)
      CALL LINEAR(TT,T,THET1,THT13,I)
      CALL LINEAR(TT,T,PRE,P3,I)
      CALL LINEAR(TT,T,VPP,VVPP3,I)
```

```
      CALL LINEAR(TT,T,RHC,RHO13,I)
      CALL LINEAR(TT,T,DT,DT13,I)
      CALL LINEAR(TT,T,DTHT,DTHT13,I)
      CALL LINEAP(TT,T,DRHO,DRHO3,I)
  523 CONTINUE
      GO TO 299
  524 IF(ITEST.EQ.1) GO TO 299
      CALL GR(1,C1,C3,A1,A2,B1,B2,THT11,TW1,T11,OMEGA,VL1,VVP1,TT,ITEST)
      DO 250 I=2,J1
      XS1=XS1+.1
      CALL GR(1,C1,C3,A1,A2,B1,B2,THT11,TW1,T11,OMEGA,VL1,VVP1,TT,ITEST)
  250 CONTINUE
      ITEST=1
      GO TO 301
  299 CALL GR(1,C1,C3,A1,A2,B1,B2,THT13,TW3,T13,OMEGA,VL3,VVP3,TT,ITEST)
      IF(TT.GT.2.625) GO TO 109
      IF(ITEST.EQ.1) GO TO 300
      DO 302 I=2,9
      B1(I)=B1(1)
      B2(I)=B2(1)
  302 CONTINUE
      GO TO 301
  300 CALL KUTTA(J,OMEGA,A1,A2,B1,B2,A1P,A2P,B1P,B2P)
  301 X=.075
      DX=.125
      DX1=.0001
      DX2=.0001
      DX3=.0001
      DX4=.0001
      ITEST=1
      DO 5 J=1,J1
      X=X+DX
      PRINT 4,J,A1(J),A2(J),B1(J),B2(J),TT,X
    4 FORMAT(5X,I5,4F15.5,2F10.6)
      GO TO 5
  525 SUM1=0.0
      SUM2=0.0
      SUM3=0.0
      SUM4=0.0
      NUSS=C1*(OMEGA(J)*(A1(J)+A2(J))
      ST=C1*C*TW3*(A1(J)+A2(J))/(RHO13*U1(J,OMEGA,VL3,VVP3)*PR)
      TAUW=C1*C*TW3*U1(J,OMEGA,VL3,VVP3)*(B1(J)+B2(J))
      CF=4.*C*TW3*(B1(J)+B2(J))/(RHO13*U1(J,OMEGA,VL3,VVP3))
      CALL SIM(FU1,A1,A2,B1,B2,DX1,SUM1)
      DEL3=SUM1
      CALL SIM(FU2,A1,A2,B1,B2,DX2,SUM2)
      DELH=SUM2
      CALL SIM(FU3,A1,A2,B1,B2,DX3,SUM3)
      DEL2=SUM3
      CALL SIM(FU4,A1,A2,B1,B2,DX4,SUM4)
      DEL1=SUM4
      PRINT,DEL3,DELH,DEL2,DEL1
    5 CONTINUE
      IF(TT.GT.TT1) GO TO 110
      GO TO 10
  109 CONTINUE
  110 CALL EXIT
      END
      SUBROUTINE KUTTA(J,X,A1,A2,B1,B2,A1P,A2P,B1P,B2P)
      IMPLICIT REAL*8(A-H),REAL*8(O-Z)
      COMMON OMEGA1,OMEGA2,P,PI,ETA,C,PR,DYE,P1,P2,P3,TW1,TW2
```

```
      COMMON TW3,T11,T12,T13,THT11,THT12,THT13,RHC0,RHC11,RHC12,RHC13
      COMMON DRHC11,DRHC12,DRHC13,VTWPX,VTHPX,DTW1,DTW2,DTW3,VL1,VL2
      COMMON VL3,DOMGA,VVP1,VVP2,VVP3,VVPP1,VVPP2,VVPP3,H,DT11,DT12,DT13
      COMMON C4,CC51,C49,CC2,CE5,CE3,CCE3,CCE5,CC5
      COMMON T1PX1,T1PX2,T1PX3,DELTA1,DELTA2,J1,CF
      COMMON DTHT11,DTHT12,DTHT13
      DIMENSION AK1(12),AK2(12),AK3(12),AK4(12),X(12),A1(12),A2(12)
      DIMENSION AL1(12),AL2(12),AL3(12),AL4(12),B1(12),B2(12),A1P(12)
      DIMENSION AM1(12),AM2(12),AM3(12),AM4(12),A2P(12),B1P(12)
      DIMENSION AN1(12),AN2(12),AN3(12),AN4(12),B2P(12)
      DIMENSION ZB1(12),ZB2(12),ZA1(12),ZA2(12)
      DO 10 J=2,J1
      A1P(J)=(A1(J)-A1(J-1))/DOMGA
      A2P(J)=(A2(J)-A2(J-1))/DOMGA
      B1P(J)=(B1(J)-B1(J-1))/DOMGA
      B2P(J)=(B2(J)-B2(J-1))/DOMGA
      D1=FM1(J,X,B1,B2,A1,A2,B1P,B2P,P1,TW1,T11,THT11,VL1,VVP1,
     1VVPP1,DTW1,DT11,RHC11,T1PX1,DRHC11)
      D2=FM2(J,X,B1,B2,A1,A2,B1P,B2P,P1,TW1,T11,THT11,VL1,VVP1,
     1VVPP1,DTHT11,DTW1,DT11,RHC11,T1PX1)
      DD1=FE1(J,X,A1,A2,B1,B2,A1P,A2P,B1P,B2P,P1,TW1,T11,THT11,
     1VL1,VVP1,VVPP1,DTHT11,DTW1,DT11,RHC11,DRHC11,T1PX1)
      DD2=FE2(J,X,A1,A2,B1,B2,A1P,A2P,B1P,B2P,P1,TW1,T11,THT11,
     1VL1,VVP1,VVPP1,DTHT11,DTW1,DT11,RHC11,DRHC11,T1PX1)
      XNUM1=D1*CC5*U1(J,X,VL1,VVP1)**2/(B2(J)**3)+D2*C4*U1(J,X,VL1,VVP1)
     1**2/(DSQRT(B1(J)**2+B2(J)**2)*(B1(J)**2+B2(J)**2))
      DEN1=C49*CC5*U1(J,X,VL1,VVP1)**4/(B1(J)*B2(J))**3-
     1C4*CC2*U1(J,X,VL1,VVP1)**4/(B1(J)**2+B2(J)**2)**3
      F1=XNUM1/DEN1
      AK1(J)=H*F1
      XNUM2=D2*C49*U1(J,X,VL1,VVP1)**2/B1(J)**3+D1*CC2*U1(J,X,VL1,VVP1)
     1**2/(DSQRT(B1(J)**2+B2(J)**2)*(B1(J)**2+B2(J)**2))
      DEN1=C49*CC5*U1(J,X,VL1,VVP1)**4/(B1(J)*B2(J))**3-
     1C4*CC2*U1(J,X,VL1,VVP1)**4/(B1(J)**2+B2(J)**2)**3
      F2=XNUM2/DEN1
      AL1(J)=H*F2
      XNUM3=DD1*CCE5*THT11**2/A2(J)**3+DD2*CE5*THT11**2/(DSQRT(A1(J)**2+
     1A2(J)**2)*(A1(J)**2+A2(J)**2))
      F3=XNUM3/DEN1
      AM1(J)=H*F3
      XNUM4=DD2*CE3*THT11**2/A1(J)**3+DD1*CCE3*THT11**2/(DSQRT(A1(J)**2+
     1A2(J)**2)*(A1(J)**2+A2(J)**2))
      F4=XNUM4/DEN1
      AN1(J)=H*F4
      ZB1(J)=B1(J)+AK1(J)/2.
      ZB2(J)=B2(J)+AL1(J)/2.
      ZA1(J)=A1(J)+AM1(J)/2.
      ZA2(J)=A2(J)+AN1(J)/2.
      D1=FM1(J,X,ZB1,ZB2,ZA1,ZA2,B1P,B2P,P2,TW2,T12,THT12,VL2,VVP2,
     1VVPP2,DTW2,DT12,RHC12,T1PX2,DRHC12)
      D2=FM2(J,X,ZB1,ZB2,ZA1,ZA2,B1P,B2P,P2,TW2,T12,THT12,VL2,VVP2,
     1VVPP2,DTHT12,DTW2,DT12,RHC12,T1PX2)
      DD1=FE1(J,X,A1,A2,B1,B2,A1P,A2P,B1P,B2P,P2,TW2,T12,THT12,
     1VL2,VVP2,VVPP2,DTHT12,DTW2,DT12,RHC12,DRHC12,T1PX2)
      DD2=FE2(J,X,A1,A2,B1,B2,A1P,A2P,B1P,B2P,P2,TW2,T12,THT12,
     1VL2,VVP2,VVPP2,DTHT12,DTW2,DT12,RHC12,DRHC12,T1PX2)
      XNUM1=D1*CC5*U1(J,X,VL2,VVP2)**2/(ZB2(J)**3)+D2*C4*U1(J,X,VL2,VVP2
     1)**2/(DSQRT(ZB1(J)**2+ZB2(J)**2)*(ZB1(J)**2+ZB2(J)**2))
      DEN1=C49*CC5*U1(J,X,VL2,VVP2)**4/(ZB1(J)*ZB2(J))**3-
     1C4*CC2*U1(J,X,VL2,VVP2)**4/(ZB1(J)**2+ZB2(J)**2)**3
      F1=XNUM1/DEN1
```

```
      AK2(J)=H*F1
      XNUM2=D2*C49*U1(J,X,VL2,VVP2)**2/ZB1(J)**3+D1*CC2*U1(J,X,VL2,VVP2)
     1**2/(DSQRT(ZB1(J)**2+ZB2(J)**2)*(ZB1(J)**2+ZB2(J)**2))
      DEN1=C49*CC5*U1(J,X,VL2,VVP2)**4/(ZB1(J)*ZB2(J))**3-
     1C4*CC2*U1(J,X,VL2,VVP2)**4/(ZB1(J)**2+ZB2(J)**2)**3
      F2=XNUM2/DEN1
      AL2(J)=H*F2
      XNUM3=DD1*CCE5*THT12**2/A2(J)**3+DD2*CE5*THT12**2/(DSQRT(A1(J)**2+
     1A2(J)**2)*(A1(J)**2+A2(J)**2))
      F3=XNUM3/DEN1
      AM2(J)=H*F3
      XNUM4=DD2*CE3*THT12**2/A1(J)**3+DD1*CCE3*THT12**2/(DSQRT(A1(J)**2+
     1A2(J)**2)*(A1(J)**2+A2(J)**2))
      F4=XNUM4/DEN1
      AN2(J)=H*F4
      ZB1(J)=B1(J)+AK2(J)/2.
      ZB2(J)=B2(J)+AL2(J)/2.
      ZA1(J)=A1(J)+AM2(J)/2.
      ZA2(J)=A2(J)+AN2(J)/2.
      D1=FM1(J,X,ZB1,ZB2,ZA1,ZA2,B1P,B2P,P2,TW2,T12,THT12,VL2,VVP2,
     1VVPP2,DTW2,DT12,RHO12,T1PX2,DRHO12)
      D2=FM2(J,X,ZB1,ZB2,ZA1,ZA2,B1P,B2P,P2,TW2,T12,THT12,VL2,VVP2,
     1VVPP2,DTHT12,DTW2,DT12,RHO12,T1PX2)
      DD1=FE1(J,X,A1,A2,B1,B2,A1P,A2P,B1P,B2P,P2,TW2,T12,THT12,
     1VL2,VVP2,VVPP2,DTHT12,DTW2,DT12,RHO12,DRHO12,T1PX2)
      DD2=FE2(J,X,A1,A2,B1,B2,A1P,A2P,B1P,B2P,P2,TW2,T12,THT12,
     1VL2,VVP2,VVPP2,DTHT12,DTW2,DT12,RHO12,DRHO12,T1PX2)
      XNUM1=D1*CC5*U1(J,X,VL2,VVP2)**2/(ZB2(J)**3)+D2*C4*U1(J,X,VL2,VVP2
     1)**2/(DSQRT(ZB1(J)**2+ZB2(J)**2)*(ZB1(J)**2+ZB2(J)**2))
      DEN1=C49*CC5*U1(J,X,VL2,VVP2)**4/(ZB1(J)*ZB2(J))**3-
     1C4*CC2*U1(J,X,VL2,VVP2)**4/(ZB1(J)**2+ZB2(J)**2)**3
      F1=XNUM1/DEN1
      AK3(J)=H*F1
      XNUM2=D2*C49*U1(J,X,VL2,VVP2)**2/ZB1(J)**3+D1*CC2*U1(J,X,VL2,VVP2)
     1**2/(DSQRT(ZB1(J)**2+ZB2(J)**2)*(ZB1(J)**2+ZB2(J)**2))
      F2=XNUM2/DEN1
      AL3(J)=H*F2
      XNUM3=DD1*CCE5*THT12**2/A2(J)**3+DD2*CE5*THT12**2/(DSQRT(A1(J)**2+
     1A2(J)**2)*(A1(J)**2+A2(J)**2))
      F3=XNUM3/DEN1
      AM3(J)=H*F3
      XNUM4=DD2*CE3*THT12**2/A1(J)**3+DD1*CCE3*THT12**2/(DSQRT(A1(J)**2+
     1A2(J)**2)*(A1(J)**2+A2(J)**2))
      F4=XNUM4/DEN1
      AN3(J)=H*F4
      ZB1(J)=B1(J)+AK3(J)
      ZB2(J)=B2(J)+AL3(J)
      ZA1(J)=A1(J)+AM3(J)
      ZA2(J)=A2(J)+AN3(J)
      D1=FM1(J,X,ZB1,ZB2,ZA1,ZA2,B1P,B2P,P3,TW3,T13,THT13,VL3,VVP3,
     1VVPP3,DTW3,DT13,RHO13,T1PX3,DRHO13)
      D2=FM2(J,X,ZB1,ZB2,ZA1,ZA2,B1P,B2P,P3,TW3,T13,THT13,VL3,VVP3,
     1VVPP3,DTHT13,DTW3,DT13,RHO13,T1PX3)
      DD1=FE1(J,X,A1,A2,B1,B2,A1P,A2P,B1P,B2P,P3,TW3,T13,THT13,
     1VL3,VVP3,VVPP3,DTHT13,DTW3,DT13,RHO13,DRHO13,T1PX3)
      DD2=FE2(J,X,A1,A2,B1,B2,A1P,A2P,B1P,B2P,P3,TW3,T13,THT13,
     1VL3,VVP3,VVPP3,DTHT13,DTW3,DT13,RHO13,DRHO13,T1PX3)
      XNUM1=D1*CC5*U1(J,X,VL3,VVP3)**2/(ZB2(J)**3)+D2*C4*U1(J,X,VL3,VVP3
     1)**2/(DSQRT(ZB1(J)**2+ZB2(J)**2)*(ZB1(J)**2+ZB2(J)**2))
      DEN1=C49*CC5*U1(J,X,VL3,VVP3)**4/(ZB1(J)*ZB2(J))**3-
     1C4*CC2*U1(J,X,VL3,VVP3)**4/(ZB1(J)**2+ZB2(J)**2)**3
```

```
      F1=XNUM1/DEN1
      AK4(J)=H*F1
      XNUM2=D2*C49*U1(J,X,VL3,VVP3)**2/ZB1(J)**3+D1*CC2*U1(J,X,VL3,VVP3)
     1**2/(DSQRT(ZB1(J)**2+ZB2(J)**2)*(ZB1(J)**2+ZB2(J)**2))
      F2=XNUM2/DEN1
      AL4(J)=H*F2
      XNUM3=DD1*CCE5*THT13**2/A2(J)**3+DD2*CE5*THT13**2/(DSQRT(A1(J)**2+
     1A2(J)**2)*(A1(J)**2+A2(J)**2))
      F3=XNUM3/DEN1
      AM4(J)=H*F3
      XNUM4=DD2*CE3*THT13**2/A1(J)**3+DD1*CCE3*THT13**2/(DSQRT(A1(J)**2+
     1A2(J)**2)*(A1(J)**2+A2(J)**2))
      F4=XNUM4/DEN1
      AN4(J)=H*F4
      B1(J)=B1(J)+(AK1(J)+2.*(AK2(J)+AK3(J))+AK4(J))/6.
      B2(J)=B2(J)+(AL1(J)+2.*(AL2(J)+AL3(J))+AL4(J))/6.
      A1(J)=A1(J)+(AM1(J)+2.*(AM2(J)+AM3(J))+AM4(J))/6.
      A2(J)=A2(J)+(AN1(J)+2.*(AN2(J)+AN3(J))+AN4(J))/6.
  100 CONTINUE
      RETURN
      END
      SUBROUTINE LINEAR(A,X,Y,VV,I)
      IMPLICIT REAL*8(A-H),REAL*8(O-Z)
      COMMON OMEGA1,OMEGA2,R,PI,ETA,C,PR,DYB,P1,P2,P3,TW1,TW2
      COMMON TW3,T11,T12,T13,THT11,THT12,THT13,RHOC,RHO11,RHO12,RHO13
      COMMON DRHO11,DRHO12,DRHO13,VTWPX,VTHPX,DTW1,DTW2,DTW3,VL1,VL2
      COMMON VL3,DOMGA,VVP1,VVP2,VVP3,VVPP1,VVPP2,VVPP3,H,DT11,DT12,DT13
      COMMON C4,CC51,C49,CC2,CE5,CE3,CCE3,CCE5,CC5
      COMMON T1PX1,T1PX2,T1PX3,DELTA1,DELTA2,J1,CP
      COMMON DTHT11,DTHT12,DTHT13
      DIMENSION X(12),Y(12)
    2 IF(A-X(I)) 3,1,1
    1 I=I+1
      GO TO 2
    3 I=I-1
      VV=Y(I)*(A-X(I+1))/(X(I)-X(I+1))+Y(I+1)*(A-X(I))/(X(I+1)-X(I))
      RETURN
      END
      SUBROUTINE NR(J,C1,C3,A1,A2,B1,B2,VTHT1,VTW,VT1,X,VL,VVP,TT,ITEST)
      IMPLICIT REAL*8(A-H),REAL*8(O-Z)
      COMMON OMEGA1,OMEGA2,R,PI,ETA,C,PR,DYB,P1,P2,P3,TW1,TW2
      COMMON TW3,T11,T12,T13,THT11,THT12,THT13,RHO0,RHO11,RHO12,RHO13
      COMMON DRHO11,DRHO12,DRHO13,VTWPX,VTHPX,DTW1,DTW2,DTW3,VL1,VL2
      COMMON VL3,DOMGA,VVP1,VVP2,VVP3,VVPP1,VVPP2,VVPP3,H,DT11,DT12,DT13
      COMMON C4,CC51,C49,CC2,CE5,CE3,CCE3,CCE5,CC5
      COMMON T1PX1,T1PX2,T1PX3,DELTA1,DELTA2,J1,CP
      COMMON DTHT11,DTHT12,DTHT13
      DIMENSION A1(12),A2(12),B1(12),B2(12),X(12),NSIGN(400)
      ETA1=.6
      YY1=.6
      II=1
      ISIGN=0
      DEL=.01
      IF(TT.GT.1.625) GO TO 636
      IF(ITEST.EQ.1) GO TO 89
   88 B2(J)=.71
   89 IF(TT.GT.1.) GO TO 689
   90 B1(J)=1./(2.*DSQRT(TT)*OMEGA1)-B2(J)*OMEGA2/OMEGA1
      F=OMEGA1*DERF(0.6D0*B1(J))+OMEGA2*DERF(0.6D0*B2(J))-DERF(.3D0/
     1DSQRT(TT))
      PRINT,B2(J),F
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```
      GO TO 120
  121 IF(ISIGN.EQ.1) GO TO 120
      IF(F) 100,120,130
  100 NSIGN(II)=0
      GO TO 131
  130 NSIGN(II)=1
  131 IF(II-1) 133,133,132
  132 IF(NSIGN(II)-NSIGN(II-1)) 120,133,120
  133 II=II+1
      B2(J)=B2(J)+DEL
      GO TO 90
  120 FP=C1*YY1*OMEGA2*(-DEXP(-B1(J)**2*YY1**2)+DEXP(-B2(J)**2*YY1**2))
      B2(J)=B2(J)-F/FP
      B1(J)=1./(2.*DSQRT(TT)*OMEGA1)-B2(J)*OMEGA2/OMEGA1
      ISIGN=1
      IF(DABS(F)-.00001) 135,135,90
  135 PRINT 136,J,B1(J),B2(J)
  136 FORMAT(5X,I3,2F15.3)
      ISIGN=0
      GO TO 636
  637 II=1
  689 IF(TT.GE.1.625) GO TO 688
      IF(TT.GT.1.5) GO TO 690
      B2(J)=0.0
      GO TO 690
  688 B2(J)=.948
  690 B1(J)=C1/OMEGA1*(.33206+(1./DSQRT(PI*TT)-.33206)*DEXP(-.25*
     1(TT-1)**2)-C1*OMEGA2*B2(J))
      F=OMEGA1*DERF(B1(J)*YY1)+OMEGA2*DERF(B2(J)*YY1)-.19894-(DERF(
     1.300/DSQRT(TT))-.19894)*DEXP(-.25*(TT-1)**2)
      PRINT,B2(J),F
      IF(TT.LE.1.5) GO TO 621
      GO TO 620
  621 IF(ISIGN.EQ.1) GO TO 620
      IF(F) 600,620,630
  600 NSIGN(II)=0
      GO TO 631
  630 NSIGN(II)=1
  631 IF(II-1) 633,633,632
  632 IF(NSIGN(II)-NSIGN(II-1)) 620,633,620
  633 II=II+1
      B2(J)=B2(J)+DEL
      GO TO 690
  620 FP=C1*YY1*OMEGA2*(-DEXP(-B1(J)**2*YY1**2)+DEXP(-B2(J)**2*YY1**2))
      B2(J)=B2(J)-F/FP
      B1(J)=C1/OMEGA1*(.33206+(1./DSQRT(PI*TT)-.33206)*DEXP(-.25*
     1(TT-1)**2)-C1*OMEGA2*B2(J))
      ISIGN=1
      IF(DABS(F)-.00001) 635,635,690
  635 PRINT 136,J,B1(J),B2(J)
  636 CONTINUE
      RETURN
      END
      SUBROUTINE SIM(F,A1,A2,B1,B2,DX,SUM)
      IMPLICIT REAL*8(A-H),REAL*8(O-Z)
      EXTERNAL F
      COMMON OMEGA1,OMEGA2,R,PI,ETA,C,PR,DYB,P1,P2,P3,TW1,TW2
      COMMON TW3,T11,T12,T13,THT11,THT12,THT13,RHO0,RHO11,RHO12,RHO13
      COMMON DRHO11,DRHO12,DRHO13,VTWPX,VTHPX,DTW1,DTW2,DTW3,VL1,VL2
      COMMON VL3,DOMGA,VVP1,VVP2,VVP3,VVPP1,VVPP2,VVPP3,H,DT11,DT12,DT13
      COMMON C4,CC51,C49,CC2,CE5,CE3,CCE3,CCE5,CC5
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      COMMON T1PX1,T1PX2,T1PX3,DELTA1,DELTA2,J1,CP
      COMMON DTHT11,DTHT12,DTHT13
      DIMENSION A1(12),A2(12),B1(12),B2(12)
   10 TEMP=0.
      SUM=0.0
      X=0.0
      NP=2
  200 SUM=SUM+4.*F(RHO1,A1,A2,B1,B2,VTHT1,VTW,VT1,P)
      IF(NP.EQ.2) GO TO 300
      IF(DABS(TEMP-SUM).LE..0005) GO TO 400
  300 TEMP=SUM
      X=X+DX
      NP=NP+1
      SUM=SUM+2.*F(RHO1,A1,A2,B1,B2,VTHT1,VTW,VT1,P)
      IF(DABS(TEMP-SUM).LE..0005) GO TO 400
      TEMP=SUM
      X=X+DX
      NP=NP+1
      GO TO 200
  400 NP2=NP/2
      IF(NP2*2.EQ.NP) GO TO 500
      X=X+DX
      NP=NP+1
      SUM=SUM+4.*F(RHO1,A1,A2,B1,B2,VTHT1,VTW,VT1,P)
  500 X=X+DX
      SUM=SUM+F(RHO1,A1,A2,B1,B2,VTHT1,VTW,VT1,P)
      XNP=NP
      SUM=X/(3.*XNP)*SUM
      PRINT,SUM
      RETURN
      END
      FUNCTION FU1(RHO1,A1,A2,B1,B2,VTHT1,VTW,VT1,P)
      IMPLICIT REAL*8(A-H),REAL*8(O-Z)
      COMMON OMEGA1,OMEGA2,R,PI,ETA,C,PR,DYB,P1,P2,P3,TW1,TW2
      COMMON TW3,T11,T12,T13,THT11,THT12,THT13,RHO0,RHO11,RHO12,RHO13
      COMMON DRHO11,DRHO12,DRHO13,VTWPX,VTHPX,DTW1,DTW2,DTW3,VL1,VL2
      COMMON VL3,DOMGA,VVP1,VVP2,VVP3,VVPP1,VVPP2,VVPP3,H,DT11,DT12,DT13
      COMMON C4,CC51,C49,CC2,CE5,CE3,CCE3,CCE5,CC5
      COMMON T1PX1,T1PX2,T1PX3,DELTA1,DELTA2,J1,CP
      COMMON DTHT11,DTHT12,DTHT13
      DIMENSION A1(12),A2(12),B1(12),B2(12)
      FU1=RHO0/RHO13*(DERF(B1(J)*X)+DERF(B2(J)*X)-(DERF(B1(J)*X)+
     2DERF(B2(J)*X))**3)
      RETURN
      END
      FUNCTION FU2(RHO1,A1,A2,B1,B2,VTHT1,VTW,VT1,P)
      IMPLICIT REAL*8(A-H),REAL*8(O-Z)
      COMMON OMEGA1,OMEGA2,R,PI,ETA,C,PR,DYB,P1,P2,P3,TW1,TW2
      COMMON TW3,T11,T12,T13,THT11,THT12,THT13,RHO0,RHO11,RHO12,RHO13
      COMMON DRHO11,DRHO12,DRHO13,VTWPX,VTHPX,DTW1,DTW2,DTW3,VL1,VL2
      COMMON VL3,DOMGA,VVP1,VVP2,VVP3,VVPP1,VVPP2,VVPP3,H,DT11,DT12,DT13
      COMMON C4,CC51,C49,CC2,CE5,CE3,CCE3,CCE5,CC5
      COMMON T1PX1,T1PX2,T1PX3,DELTA1,DELTA2,J1,CP
      COMMON DTHT11,DTHT12,DTHT13
      DIMENSION A1(12),A2(12),B1(12),B2(12)
      FU2=RHO0/RHO13*((DERF(B1(J)*X)+DERF(B2(J)*X))*(VTHT1*(DERF(A1(J)*X
     1)+DERF(A2(J)*X))/VT1+VTW/VT1-1.))
      RETURN
      END
      FUNCTION FU3(RHO1,A1,A2,B1,B2,VTHT1,VTW,VT1,P)
      IMPLICIT REAL*8(A-H),REAL*8(O-Z)
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      COMMON OMEGA1,OMEGA2,R,PI,ETA,C,PR,DYB,P1,P2,P3,TW1,TW2
      COMMON TW3,T11,T12,T13,THT11,THT12,THT13,RHO0,RHO11,RHO12,RHO13
      COMMON DRHO11,DRHO12,DRHO13,VTWPX,VTHPX,DTW1,DTW2,DTW3,VL1,VL2
      COMMON VL3,DOMGA,VVP1,VVP2,VVP3,VVPP1,VVPP2,VVPP3,H,DT11,DT12,DT13
      COMMON C4,CC51,C49,CC2,CE5,CE3,CCE3,CCE5,CC5
      COMMON T1PX1,T1PX2,T1PX3,DELTA1,DELTA2,J1,CP
      COMMON DTHT11,DTHT12,DTHT13
      DIMENSION A1(12),A2(12),B1(12),B2(12)
      FU3=RHO0/RHO13*(DERF(B1(J)*X)+DERF(B2(J)*X)-(DERF(B1(J)*X)+
     1DERF(B2(J)*X))**2)
      RETURN
      END
      FUNCTION FU4(RHO1,A1,A2,B1,B2,VTHT1,VTW,VT1,P)
      IMPLICIT REAL*8(A-H),REAL*8(O-Z)
      COMMON OMEGA1,OMEGA2,R,PI,ETA,C,PR,DYB,P1,P2,P3,TW1,TW2
      COMMON TW3,T11,T12,T13,THT11,THT12,THT13,RHO0,RHO11,RHO12,RHO13
      COMMON DRHO11,DRHO12,DRHO13,VTWPX,VTHPX,DTW1,DTW2,DTW3,VL1,VL2
      COMMON VL3,DOMGA,VVP1,VVP2,VVP3,VVPP1,VVPP2,VVPP3,H,DT11,DT12,DT13
      COMMON C4,CC51,C49,CC2,CE5,CE3,CCE3,CCE5,CC5
      COMMON T1PX1,T1PX2,T1PX3,DELTA1,DELTA2,J1,CP
      COMMON DTHT11,DTHT12,DTHT13
      DIMENSION A1(12),A2(12),B1(12),B2(12)
      FU4=RHO0*R*VTHT1/P*(DERF(A1(J)*X)+DERF(A2(J)*X))+RHO *(R*VTHT1/P+
     1ETA)-RHO0/RHO13*(DERF(B1(J)*X)+DERF(B2(J)*X))
      RETURN
      END
      FUNCTION FM1(J,X,B1,B2,A1,A2,B1P,B2P,P,VTW,VT1,VTHT1,VL,VVP,VVPP,
     1FTW,DT1,RHO1,T1PX,DRHO1)
      IMPLICIT REAL*8(A-H),REAL*8(O-Z)
      COMMON OMEGA1,OMEGA2,R,PI,ETA,C,PR,DYB,P1,P2,P3,TW1,TW2
      COMMON TW3,T11,T12,T13,THT11,THT12,THT13,RHO0,RHO11,RHO12,RHO13
      COMMON DRHO11,DRHO12,DRHO13,VTWPX,VTHPX,DTW1,DTW2,DTW3,VL1,VL2
      COMMON VL3,DOMGA,VVP1,VVP2,VVP3,VVPP1,VVPP2,VVPP3,H,DT11,DT12,DT13
      COMMON C4,CC51,C49,CC2,CE5,CE3,CCE3,CCE5,CC5
      COMMON T1PX1,T1PX2,T1PX3,DELTA1,DELTA2,J1,CP
      COMMON DTHT11,DTHT12,DTHT13
      DIMENSION X(12),B1(12),B2(12),A1(12),A2(12),B1P(12),B2P(12)
      C1=-.5*DSQRT(2./PI)*OMEGA1**2
      C2=-OMEGA1**2/PI
      C3=-1./DSQRT(PI)*OMEGA1*OMEGA2
      C4=C3
      C5=-2.*OMEGA1*OMEGA2/PI
      C6=-4./(PI*DSQRT(PI))*OMEGA1**3
      C7=-1./(PI*DSQRT(PI))*OMEGA1**3
      C8=-4.*OMEGA1**2*OMEGA2/(PI*DSQRT(PI))
      C9=-2.*OMEGA1**2*OMEGA2/(PI*DSQRT(PI))
      C10=C9
      C11=-2.*OMEGA1*OMEGA2**2/(PI*DSQRT(PI))
      C12=-2.*OMEGA1*OMEGA2**2/(PI*DSQRT(PI))
      C13=2.*OMEGA1**3/(PI*DSQRT(PI))
      C14=OMEGA1**2*OMEGA2/(PI*DSQRT(PI))
      C15=(1./3.)*C13
      C16=-C15
      C17=-C8
      C18=2.*OMEGA1**2*OMEGA2/(PI*DSQRT(PI))
      C19=C9
      C20=C8
      C21=C7
      C22=-C7
      C23=-OMEGA1**2*OMEGA2/(PI*DSQRT(PI))
      C24=OMEGA1**2*OMEGA2/(PI*DSQRT(PI))
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      C25=2.*OMEGA1**2*OMEGA2/(PI*DSQRT(PI))
      C26=2.*OMEGA1*OMEGA2**2/(PI*DSQRT(PI))
      C27=C17
      C28=2.*OMEGA1**2*OMEGA2/(PI*DSQRT(PI))
      C29=C9
      C30=C8
      C31=4.*OMEGA1*OMEGA2**2/(PI*DSQRT(PI))
      C32=.5*C31
      C33=-C32
      C34=2.*C33
      C35=C9
      C36=2.*OMEGA1**2*OMEGA2/(PI*DSQRT(PI))
      C37=C33
      C38=C32
      C39=-C2
      C40=-C5
      C41=OMEGA1*DELTA1/DSQRT(PI)
      C42=OMEGA1*DELTA2/DSQRT(PI)
      C43=OMEGA1/DSQRT(PI)
      C44=C41
      C45=C42/DSQRT(PI)
      C46=C43
      C47=4.*OMEGA1**2/PI
      C48=4.*OMEGA1*OMEGA2/PI
      C49=OMEGA1**2*DSQRT(2./PI)/4.
      FM11=C1*U1(J,X,VL,VVP)*DU1(J,X,VL,VVPP,VVP)/(B1(J)*B1(J))+C2*
     1U1(J,X,VL,VVP)*U1(J,X,VL,VVP)*DYB/B1(J)+C3*U1(J,X,VL,VVP)*DU1(
     2J,X,VL,VVPP,VVP)*B2(J)/(B1(J)*B1(J)*DSQRT(B1(J)*B1(J)+B2(J)*B2(J))
     3)
      FM12=C5*U1(J,X,VL,VVP)*U1(J,X,VL,VVP)*DYB*B2(J)/(B1(J)*B1(J)+
     1B2(J)*B2(J))+C6*U1(J,X,VL,VVP)*U1(J,X,VL,VVP)*U1PX(VVP,VL)*DATAN(
     2.707D0)/(1.414*B1(J)*B1(J))+C7*U1(J,X,VL,VVP)**3*B1P(J)*
     3((1./(1.414*B1(J)**3))*DATAN(.707D0)+1./(3.*B1(J)**3))
      FM13=C8*U1(J,X,VL,VVP)**2*U1PX(VVP,VL)*(DATAN(B2(J)/(1.414*B1(J)
     1))/(1.414*B1(J)**2)+B2(J)*DATAN(B1(J)/DSQRT(B1(J)**2+B2(J)**2))/
     2(B1(J)**2*DSQRT(B1(J)**2+B2(J)**2)))+C9*U1(J,X,VL,VVP)**3*B2P(J)*
     3(DATAN(B1(J)/DSQRT(B1(J)**2+B2(J)**2))/((DSQRT(B1(J)**2+B2(J)**2))
     4**3)+B1(J)/((B1(J)**2+B2(J)**2)*(2.*B1(J)**2+B2(J)**2)))
      FM14=C10*U1(J,X,VL,VVP)**3*B1P(J)*(DATAN(B2(J)/(1.414*B1(J)))/(
     12.*1.414*B1(J)**3)+B2(J)/((2.*B1(J)**2)*(B2(J)**2+2.*B1(J)**2)))+
     2C11*U1(J,X,VL,VVP)**2*U1PX(VVP,VL)*(2.*B2(J)/(B1(J)**2*PI)*DATAN(
     3B2(J)/DSQRT(B1(J)**2+B2(J)**2))/DSQRT(B1(J)**2+B2(J)**2))
      FM15=C12*U1(J,X,VL,VVP)**3*B2P(J)*(DATAN(B2(J)/DSQRT(B1(J)**2+
     1B2(J)**2))/(DSQRT(B1(J)**2+B2(J)**2))**3+B2(J)/((B1(J)**2+B2(J)
     2**2)*(B1(J)**2+2.*B2(J)**2)))+C13*U1(J,X,VL,VVP)**2*U1PX(VVP,VL)
     3*B1(J)*(DATAN(0.707D0)/(2.*1.414*B1(J)**3)+1./(6.*B1(J)**3))
      FM16=C14*U1(J,X,VL,VVP)**2*U1PX(VVP,VL)*B1(J)*(DATAN(B2(J)/(1.414*
     1B1(J)))/(1.414*B1(J)**3)+B2(J)/(B1(J)**2*(B2(J)**2+2.*B1(J)**2
     2)))+C15*U1(J,X,VL,VVP)**2*U1PX(VVP,VL)/(B1(J)**2)+C16*U1(J,X,VL,VV
     3P)**3*B1P(J)/(B1(J)**3)+C17*U1(J,X,VL,VVP)**3*B1(J)*B2P(J)/
     4(2.*B1(J)**2+B2(J)**2)**2
      FM17=C18*U1(J,X,VL,VVP)**2*U1PX(VVP,VL)*B1(J)/(B2(J)*(2.*B1(J)**2
     1+B2(J)**2))+C19*U1(J,X,VL,VVP)**3*B1(J)*B2P(J)/(B2(J)**2*(2.*
     2B1(J)**2+B2(J)**2))+C20*U1(J,X,VL,VVP)**3*B1(J)*B2P(J)/((2.*B1(J)
     3**2+B2(J)**2)**2)+C21*U1(J,X,VL,VVP)**2*U1PX(VVP,VL)/(B1(J)**2)
      FM18=C22*U1(J,X,VL,VVP)**3*B1P(J)/(B1(J)**3)+C23*U1(J,X,VL,VVP)**2
     1*U1PX(VVP,VL)/(B2(J)*B1(J))+C24*U1(J,X,VL,VVP)**3*B2P(J)/(B1(J)*
     2B2(J)**2)+C25*U1(J,X,VL,VVP)**2*B2(J)*U1PX(VVP,VL)*(DATAN(B1(J)
     3/DSQRT(B1(J)**2+B2(J)**2))/((DSQRT(B1(J)**2+B2(J)**2))**3)
     4+B1(J)/((B1(J)**2+B2(J)**2)*(2.*B1(J)**2+B2(J)**2)))
      FM19=C26*U1(J,X,VL,VVP)**2*U1PX(VVP,VL)*B2(J)*(DATAN(B2(J)/DSQRT(
```

```
1B1(J)**2+B2(J)**2))/((DSQRT(B1(J)**2+B2(J)**2))**3)+B2(J)/((B1(J)
2**2+B2(J)**2)*(B1(J)**2+2.*B2(J)**2)))+C27*U1(J,X,VL,VVP)**3*B2(J)
3*B1P(J)/((2.*B1(J)**2+B2(J)**2)**2)+C28*U1(J,X,VL,VVP)**2*U1PX(
4VVP,VL)*B2(J)/(B1(J)*(2.*B1(J)**2+B2(J)**2))+C29*U1(J,X,VL,VVP)
5**3*B2(J)*B1P(J)/(B1(J)**2*(2.*B1(J)**2+B2(J)**2))
 FM111=C30*U1(J,X,VL,VVP)**3*B2(J)*B1P(J)/((2.*B1(J)**2+B2(J)**2)
1**2)+C31*U1(J,X,VL,VVP)**3*B2(J)*B2P(J)/((B1(J)**2+2.*B2(J)**2)**2
2)+C32*U1(J,X,VL,VVP)**2*U1PX(VVP,VL)/(B1(J)**2+2.*B2(J)**2)+C33*
3U1(J,X,VL,VVP)**3*B2P(J)/(B2(J)*(B1(J)**2+2.*B2(J)**2))+C34*
4U1(J,X,VL,VVP)**3*B2(J)*B2P(J)/((B1(J)**2+2.*B2(J)**2)**2)
 FM112=C35*U1(J,X,VL,VVP)**2*U1PX(VVP,VL)*B2(J)/(B1(J)*(B1(J)**2+
1B2(J)**2))+C36*U1(J,X,VL,VVP)**3*B2(J)*B1P(J)/(B1(J)**2*(B1(J)**2
2+B2(J)**2))+C37*U1(J,X,VL,VVP)**2*U1PX(VVP,VL)/(B1(J)**2+B2(J)**2)
3+C38*U1(J,X,VL,VVP)**3*B2P(J)/(B2(J)*(B1(J)**2+B2(J)**2))+C39*
4U1(J,X,VL,VVP)**2*DYB/B1(J)
 FM113=C40*U1(J,X,VL,VVP)**2*B2(J)*DYB/(B1(J)**2+B2(J)**2)+C41*
1U1(J,X,VL,VVP)*DU1(J,X,VL,VVPP,VVP)*R*VTHT1*A1(J)/((R*VTHT1+ETA*P+
2R*VTW)*B1(J)**2*DSQRT(A1(J)**2+B1(J)**2))+C42*U1(J,X,VL,VVP)*
3DU1(J,X,VL,VVPP,VVP)*R*VTHT1*A2(J)/((R*VTHT1+ETA*P+R*VTW)*B1(J)**2
4*DSQRT(A2(J)**2+B1(J)**2))
 FM114=C43*U1(J,X,VL,VVP)*DU1(J,X,VL,VVPP,VVP)*(ETA*P+R*VTW)/((
1R*VTHT1+ETA*P+R*VTW)*B1(J)**2)+C44*U1(J,X,VL,VVP)**2*U1PX(VVP,VL)
2*R*VTHT1*A1(J)/((R*VTHT1+ETA*P+R*VTW)*B1(J)**2*DSQRT(A1(J)**2+
3B1(J)**2))+C45*U1(J,X,VL,VVP)**2*U1PX(VVP,VL)*R*VTHT1*A2(J)/((R*
4VTHT1+ETA*P+R*VTW)*B1(J)**2*DSQRT(A2(J)**2+B1(J)**2))
 FM06=2.*C*ETA*VTHT1/DSQRT(PI)*(A1(J)/(2.*(A1(J)**2+2.*B1(J)**2))+
1A2(J)/(2.*(A2(J)**2+2.*B1(J)**2)))
 FM07=VTHT1*(DATAN(A1(J)/(1.414*B1(J)))/(4.*DSQRT(2.*PI)*B1(J)**3)+
1A1(J)/(4.*DSQRT(PI)*B1(J)**2*(A1(J)**2+2.*B1(J)**2)))
 FM07=-2.*B1(J)**2*C*ETA*FM07
 FM08=VTHT1*(DATAN(A2(J)/(1.414*B1(J)))/(4.*DSQRT(2.*PI)*B1(J)**3)+
1A2(J)/(4.*DSQRT(PI)*B1(J)**2*(A2(J)**2+2.*B1(J)**2)))+DSQRT(2.*PI)
2*VTW/(16.*B1(J)**3)
 FM08=-2.*B1(J)**2*C*ETA*FM08
 X1=(A1(J)**2*(2.*A1(J)**2+6.*B1(J)**2))/(16.*B1(J)**4*(DSQRT(2.*
1B1(J)**2+A1(J)**2)**3))
 X2=-A1(J)**3/3.*(A1(J)*(60.*B1(J)**4+40.*(A1(J)*B1(J))**2+8.*A1(J)
2**4)/(64.*B1(J)**6*DSQRT(2.*B1(J)**2+A1(J)**2)**5))
 X3=A1(J)**5/10.*(A1(J)*(105.*(2.*B1(J)**2)**3+210.*(2.*B1(J)**2*
1A1(J))**2+168.*(2.*B1(J)**2)*A1(J)**4+48.*A1(J)**6)/(16.*(2.*B1(J)
2**2)**4*DSQRT(2.*B1(J)**2+A1(J)**2)**7))
 X4=-A1(J)**7/42.*(A1(J)*(945.*(2.*B1(J)**2)**4+2520.*(2.*B1(J)**2)
1**3*A1(J)**2+3024.*(2.*B1(J)**2)**2*A1(J)**4+1728.*(2.*B1(J)**2)
2*A1(J)**6+384.*A1(J)**8)/(32.*(2.*B1(J)**2)**5*DSQRT(A1(J)**2+
32.*B1(J)**2)**9))
 FM517=1.16*(X1+X2+X3+X4)
 FM01=VTHT1**2*FM517
 XX1=(A1(J)*A2(J)*(2.*A2(J)**2+6.*B1(J)**2))/(4.*(2.*B1(J)**2)**2*
1DSQRT(2.*B1(J)**2+A2(J)**2)**3)
 XX2=-A1(J)**3/3.*(A2(J)*(15.*(2.*B1(J)**2)**2+20.*(2.*B1(J)**2)*
1A2(J)**2+8.*A2(J)**4)/(8.*(2.*B1(J)**2)**3*DSQRT(2.*B1(J)**2+A2(J)
2**2)**5))
 XX3=A1(J)**5/10.*(A2(J)*(105.*(2.*B1(J)**2)**3+210.*(2.*B1(J)**2)
1**2*A2(J)**2+168.*(2.*B1(J)**2)*A2(J)**4+48.*A2(J)**6)/(16.*(2.*
2B1(J)**2)**4*DSQRT(2.*B1(J)**2+A2(J)**2)**7))
 XX4=-A1(J)**7/42.*(A2(J)*(945.*(2.*B1(J)**2)**4+2520.*(2.*B1(J)
1**2)**3*A2(J)**2+1728.*2.*B1(J)**2*A2(J)**6+384.*A2(J)**8+3024.*
2(2.*B1(J)**2)**2*A2(J)**4)/(32.*(2.*B1(J)**2)**5*DSQRT(2.*
3B1(J)**2+A2(J)**2)**9))
 FM518=1.16*(XX1+XX2+XX3+XX4)
 FM518=2.*VTHT1**2*FM518
```

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 XXX1=(A2(J)**2*(2.*A2(J)**2+6.*B1(J)**2))/(4.*(2.*B1(J)**2)**2*
2DSQRT(2.*B1(J)**2+A2(J)**2)**3)
 XXX2=-A2(J)**3/3.*(A2(J)*(15.*(2.*B1(J)**2)**2+20.*2.*B1(J)**2*
1A2(J)**2+8.*A2(J)**4)/(8.*(2.*B1(J)**2)**3*DSQRT(2.*B1(J)**2+A2(J)
2**2)**5))
 XXX3=A2(J)**5/10.*(A2(J)*(105.*(2.*B1(J)**2)**3+210.*(2.*B1(J)**2)
1**2*A2(J)**2+168.*2.*B1(J)**2*A2(J)**4+48.*A2(J)**6)/(16.*(2.*
2B1(J)**2)**4*DSQRT(2.*B1(J)**2+A2(J)**2)**7))
 XXX4=-A2(J)**7/42.*(A2(J)*(945*(2.*B1(J)**2)**4+2520.*(2.*B1(J)**
12)**3*A2(J)**2+3024.*(2.*B1(J)**2)**2*A2(J)**4+1728.*2.*B1(J)**2*
2A2(J)**6+384.*A2(J)**8)/(32.*(2.*B1(J)**2)**5*DSQRT(2.*B1(J)**2+
3A2(J)**2)**9))
 FM519=1.16*(XXX1+XXX2+XXX3+XXX4)
 FM02=VTHT1**2*FM519
 FM03=2.*VTHT1*VTW*(DATAN(A1(J)/(1.414*B1(J)))/(4.*DSQRT(2.*PI)*
1B1(J)**3)+A1(J)/(4.*DSQRT(PI)*B1(J)**2*(A1(J)**2+2.*B1(J)**2)))
 FM04=2.*VTHT1*VTW*(DATAN(A2(J)/(1.414*B1(J)))/(4.*DSQRT(2.*PI)
1*B1(J)**3)+A2(J)/(4.*DSQRT(PI)*B1(J)**2*(A2(J)**2+2.*B1(J)**2)))
 FM05=DSQRT(2.*PI)*VTW**2/(16.*B1(J)**3)
 FM=-2.*R*C*B1(J)**2/P*(FM01+FM02+FM03+FM04+FM05+FM518)
 FM001=.5*DSQRT(2./PI)*A1(J)**2*VTHT1**2/(DSQRT(A1(J)**2+B1(J)**2)*
1(A1(J)**2+2.*B1(J)**2))
 FM002=1./DSQRT(PI)*A1(J)*A2(J)*VTHT1**2/(DSQRT(A1(J)**2+A2(J)**2+2
1.*B1(J)**2)*(A2(J)**2+2.*B1(J)**2))
 FM003=1./DSQRT(PI)*A1(J)*A2(J)*VTHT1**2/(DSQRT(A1(J)**2+A2(J)**2+2
1.*B1(J)**2)*(A1(J)**2+2.*B1(J)**2))
 FM004=.5*DSQRT(2./PI)*A2(J)**2*VTHT1**2/(DSQRT(A2(J)**2+B1(J)**2)*
1(A2(J)**2+2.*B1(J)**2))
 FM005=1./DSQRT(PI)*VTHT1*VTW*A1(J)/(A1(J)**2+2.*B1(J)**2)
 FM006=1./DSQRT(PI)*VTHT1*VTW*A2(J)/(A2(J)**2+2.*B1(J)**2)
 FM09=2.*R*C/P*(FM001+FM002+FM003+FM004+FM005+FM006)
 FM51=FM09+FM+FM06+FM07+FM08
 FFM1=1./DSQRT(PI)*A1(J)**2*VTHT1**2/(DSQRT(2.*A1(J)**2+B1(J)**2+
1B2(J)**2)*(A1(J)**2+B1(J)**2+B2(J)**2))
 FFM2=1./DSQRT(PI)*A1(J)*A2(J)*VTHT1**2/(DSQRT(A1(J)**2+A2(J)**2+
1B1(J)**2+B2(J)**2)*(A2(J)**2+B1(J)**2+B2(J)**2))
 FFM3=1./DSQRT(PI)*A1(J)*A2(J)*VTHT1**2/(DSQRT(A1(J)**2+A2(J)**2+
1B1(J)**2+B2(J)**2)*(A1(J)**2+B1(J)**2+B2(J)**2))
 FFM4=VTHT1**2*A2(J)**2/(DSQRT(PI)*DSQRT(2.*A2(J)**2+B1(J)**2+
1B2(J)**2)*(A2(J)**2+B1(J)**2+B2(J)**2))
 FFM5=1./DSQRT(PI)*A1(J)*VTW*VTHT1/(A1(J)**2+B1(J)**2+B2(J)**2)
 FFM6=1./DSQRT(PI)*A2(J)*VTW*VTHT1/(A2(J)**2+B1(J)**2+B2(J)**2)
 F1=2.*R*C/P*(FFM1+FFM2+FFM3+FFM4+FFM5+FFM6)
 Z1=(A1(J)**2*(2.*A1(J)**2+3.*(B1(J)**2+B2(J)**2)))/(4.*(B1(J)**2+
1B2(J)**2)**2*DSQRT(A1(J)**2+B1(J)**2+B2(J)**2)**3)
 Z2=-A1(J)**3/3.*(A1(J)*(15.*(B1(J)**2+B2(J)**2)**2+20.*A1(J)**2*
1(B1(J)**2+B2(J)**2)+8.*A1(J)**4)/(8.*(B1(J)**2+B2(J)**2)**3*DSQRT(
2B1(J)**2+B2(J)**2+A1(J)**2)**5))
 Z3=A1(J)**5/10.*(A1(J)*(105.*(B1(J)**2+B2(J)**2)**3+210.*A1(J)**2*
1(B1(J)**2+B2(J)**2)**2+168.*A1(J)**4*(B1(J)**2+B2(J)**2)+48.*
2A1(J)**6)/(16.*(B1(J)**2+B2(J)**2)**4*DSQRT(A1(J)**2+B1(J)**2+
3B2(J)**2)**7))
 Z4=-A1(J)**7/42.*(A1(J)*(945.*(B1(J)**2+B2(J)**2)**4+2520.*A1(J)
1**2*(B1(J)**2+B2(J)**2)**3+3024.*A1(J)**4*(B1(J)**2+B2(J)**2)**2+
21728.*(B1(J)**2+B2(J)**2)*A1(J)**6+384.*A1(J)**8)/(32.*(B1(J)**2+
3B2(J)**2)**5*DSQRT(A1(J)**2+B1(J)**2+B2(J)**2)**9))
 FM527=1.16*(Z1+Z2+Z3+Z4)
 FFM7=VTHT1**2*FM527
 ZZ1=(A1(J)*A2(J)*(2.*A2(J)**2+3.*(B1(J)**2+B2(J)**2)))/(4.*(B1(J)*
1*2+B2(J)**2)**2*DSQRT(A2(J)**2+B1(J)**2+B2(J)**2)**3)
 ZZ2=-A1(J)**3/3.*(A2(J)*(15.*(B1(J)**2+B2(J)**2)**2+20.*A2(J)**2*(
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1B1(J)**2+B2(J)**2)+8.*A2(J)**4)/(8.*(B1(J)**2+B2(J)**2)**3*DSQRT(
2A2(J)**2+B1(J)**2+B2(J)**2)**5))
 ZZ3=A1(J)**5/16.*(A2(J)*(1.5.*(B1(J)**2+B2(J)**2)**3+210.*A2(J)**2
1*(B1(J)**2+B2(J)**2)**2+168.*A2(J)**4*(B1(J)**2+B2(J)**2)+48.*
2A2(J)**6)/(16.*(B1(J)**2+B2(J)**2)**4*DSQRT(A2(J)**2+B1(J)**2+
3B2(J)**2)**7))
 ZZ4=-A1(J)**7/42.*(A2(J)*(945.*(B1(J)**2+B2(J)**2)**4+252 .*A2(J)
1**2*(B1(J)**2+B2(J)**2)**3+3024.*A2(J)**4*(B1(J)**2+B2(J)**2)**2+
21728.*A2(J)**6*(B1(J)**2+B2(J)**2)+384.*A2(J)**8)/(32.*(B1(J)**2+
3B2(J)**2)**5*DSQRT(A2(J)**2+B1(J)**2+B2(J)**2)**9))
 FM528=1.16*(ZZ1+ZZ2+ZZ3+ZZ4)
 FM528=VTHT1**2*FM528
 ZZZ1=(A2(J)**2*(2.*A2(J)**2+3.*(B1(J)**2+B2(J)**2)))/(4.*(B1(J)**2
1+B2(J)**2)**2*DSQRT(A2(J)**2+B1(J)**2+B2(J)**2)**3)
 ZZZ2=-A2(J)**3/3.*(A2(J)*(15.*(B1(J)**2+B2(J)**2)**2+20.*A2(J)**2*
1(B1(J)**2+B2(J)**2)+8.*A2(J)**4)/(8.*(B1(J)**2+B2(J)**2)**3*DSQRT(
2A2(J)**2+B1(J)**2+B2(J)**2)**5))
 ZZZ3=A2(J)**5/15.*(A2(J)*(1.5.*(B1(J)**2+B2(J)**2)**3+210.*A2(J)
1**2*(B1(J)**2+B2(J)**2)**2+168.*A2(J)**4*(B1(J)**2+B2(J)**2)+48.*
2A2(J)**6)/(16.*(B1(J)**2+B2(J)**2)**4*DSQRT(A2(J)**2+B1(J)**2+
3B2(J)**2)**7))
 ZZZ4=-A2(J)**7/42.*(A2(J)*(945.*(B1(J)**2+B2(J)**2)**4+2520.*A2(J)
1**2*(B1(J)**2+B2(J)**2)**3+3.24.*A2(J)**4*(B1(J)**2+B2(J)**2)**2+
21728.*A2(J)**6*(B1(J)**2+B2(J)**2)+384.*A2(J)**8)/(32.*(B1(J)**2+
3B2(J)**2)**5*DSQRT(A2(J)**2+B1(J)**2+B2(J)**2)**9))
 FM529=1.16*(ZZZ1+ZZZ2+ZZZ3+ZZZ4)
 FM529=VTHT1**2*FM529
 FFM10=2.*VTHT1*VTW*(DATAN(A1(J)/DSQRT(B1(J)**2+B2(J)**2))/(2.*
1DSQRT(PI)*(DSQRT(B1(J)**2+B2(J)**2))**3)+A1(J)/(2.*DSQRT(PI)*
2(B1(J)**2+B2(J)**2)*(A1(J)**2+B1(J)**2+B2(J)**2)))
 FFM11=2.*VTHT1*VTW*(DATAN(A2(J)/DSQRT(B1(J)**2+B2(J)**2))/(2.*
1DSQRT(PI)*(DSQRT(B1(J)**2+B2(J)**2))**3)+A2(J)/(2.*DSQRT(PI)*(
2B1(J)**2+B2(J)**2)*(A2(J)**2+B1(J)**2+B2(J)**2)))+.25*DSQRT(PI)*
3VTW**2/(DSQRT(B1(J)**2+B2(J)**2)*(B1(J)**2+B2(J)**2))
 FFM8=-2.*R*C*B2(J)**2/P*(FFM7+FM528+FM529+FFM10+FFM11)
 FFM13=C*ETA*(VTHT1/DSQRT(PI))*(A1(J)/(A1(J)**2+B1(J)**2+B2(J)**2
1)+A2(J)/(A2(J)**2+B1(J)**2+B2(J)**2))
 FFM15=VTHT1*(DATAN(A1(J)/DSQRT(B1(J)**2+B2(J)**2))/(2.*DSQRT(PI)
1*(DSQRT(B1(J)**2+B2(J)**2))**3)+A1(J)/(2.*DSQRT(PI)*(B1(J)**2+B2(J
2)**2)*(A1(J)**2+B1(J)**2+B2(J)**2)))
 FFM16=VTHT1/(2.*DSQRT(PI))*(DATAN(A2(J)/DSQRT(B1(J)**2+B2(J)**2)
1)/((DSQRT(B1(J)**2+B2(J)**2))**3)+A2(J)/((B1(J)**2+B2(J)**2)*(
2A2(J)**2+B1(J)**2+B2(J)**2)))+.25*DSQRT(PI)*VTW/(DSQRT(B1(J)**2+
3B2(J)**2)*(B1(J)**2+B2(J)**2))
 FFM14=-2.*C*ETA*B2(J)**2*(FFM15+FFM16)
 FM52=F1+FFM8+FFM13+FFM14
 FM115=C46*U1(J,X,VL,VVP)**2*U1PX(VVP,VL)*(ETA*P+R*VTW)/((R*VTHT1+
1ETA*P+R*VTW)*B1(J)**2)+C47*FM51+C48*FM52
 FM1=(FM11+FM12+FM13+FM14+FM15+FM16+FM17+FM18+FM19+FM111+FM112+
 RETURN
 END
 FUNCTION FM2(J,X,B1,B2,A1,A2,B1P,B2P,P,VTW,VT1,VTHT1,VL,VVP,VVPP,
1DTHT1,DTW,DT1,RHO1,T1PX)
 IMPLICIT REAL*8(A-H),REAL*8(O-Z)
 COMMON OMEGA1,OMEGA2,R,PI,ETA,C,PR,DYB,P1,P2,P3,TW1,TW2
 COMMON TW3,T11,T12,T13,THT11,THT12,THT13,RHO0,RHO11,RHO12,RHO13
 COMMON DRHO11,DRHO12,DRHO13,VTWPX,VTHPX,DTW1,DTW2,DTW3,VL1,VL2
 COMMON VL3,DOMGA,VVP1,VVP2,VVP3,VVPP1,VVPP2,VVPP3,H,DT11,DT12,DT13
 COMMON C4,CC51,C49,CC2,CE5,CE3,CCE3,CCE5,CC5
 COMMON T1PX1,T1PX2,T1PX3,DELTA1,DELTA2,J1,CP
 COMMON DTHT11,DTHT12,DTHT13
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```
      DIMENSION X(12),B1(12),B2(12),A1(12),A2(12),B1P(12),B2P(12)
      CC1=-OMEGA1*OMEGA2/DSQRT(PI)
      CC2=CC1
      CC3=-2.*OMEGA1*OMEGA2/PI
      CC4=-.5*DSQRT(2./PI)*OMEGA2**2
      CC5=-.5*CC4
      CC6=-OMEGA2**2/PI
      CC7=-4.*OMEGA1**2*OMEGA2/(PI*DSQRT(PI))
      CC8=.5*CC7
      CC9=-4.*OMEGA1*OMEGA2**2/(PI*DSQRT(PI))
      CC10=.5*CC9
      CC11=CC10
      CC12=-4.*OMEGA2**3/(PI*DSQRT(2.*PI))
      CC13=-2.*OMEGA2**3/(PI*DSQRT(PI))
      CC14=2.*OMEGA1**2*OMEGA2/(PI*DSQRT(PI))
      CC15=-CC14
      CC16=4.*OMEGA1**2*OMEGA2**3/(PI*DSQRT(PI))
      CC17=2.*OMEGA1**2*OMEGA2/(PI*DSQRT(PI))
      CC18=-CC17
      CC19=2.*CC18
      CC20=-CC9
      CC21=.5*CC20
      CC22=-CC21
      CC23=-CC20
      CC24=CC18
      CC25=-CC24
      CC26=CC22
      CC27=CC21
      CC28=CC21
      CC29=2.*OMEGA2**3/(PI*DSQRT(PI))
      CC30=CC20
      CC31=CC28
      CC32=CC24
      CC33=CC19
      CC34=4.*OMEGA2**3/(9.*PI*DSQRT(PI))
      CC35=2.*OMEGA2**3/(3.*PI*DSQRT(PI))
      CC36=-CC35
      CC37=-CC34
      CC38=-OMEGA1*OMEGA2**2/(PI*DSQRT(PI))
      CC39=-CC38
      CC40=-OMEGA2**3/(PI*DSQRT(PI))
      CC41=-CC40
      CC42=2.*OMEGA1*OMEGA2/PI
      CC43=OMEGA2**2/PI
      CC44=DELTA1*OMEGA2/DSQRT(PI)
      CC45=DELTA2*OMEGA2/DSQRT(PI)
      CC46=OMEGA2/DSQRT(PI)
      CC47=DELTA1*OMEGA2/DSQRT(PI)
      CC48=CC45
      CC49=CC46
      CC50=4.*OMEGA1*OMEGA2/PI
      CC51=4.*OMEGA2**2/PI
      FM01=CC1*U1(J,X,VL,VVP)*DU1(J,X,VL,VVPP,VVP)*B1(J)/(B2(J)**2*
     1DSQRT(B1(J)**2+B2(J)**2))
      FM03=CC3*U1(J,X,VL,VVP)**2*B1(J)*DYB/(B1(J)**2+B2(J)**2)
      FM04=CC4*U1(J,X,VL,VVP)*DU1(J,X,VL,VVPP,VVP)/(B2(J)**2)
      FM05=CC6*U1(J,X,VL,VVP)**2*DYB/B2(J)
      FM06=CC7*U1(J,X,VL,VVP)**2*U1PX(VVP,VL)*B1(J)*DATAN(B1(J)/DSQRT(
     1B1(J)**2+B2(J)**2))/(B2(J)**2*DSQRT(B1(J)**2+B2(J)**2))
      FM07=CC8*U1(J,X,VL,VVP)**3*B1P(J)*(DATAN(B1(J)/DSQRT(B1(J)**2+B2(J
     1)**2))/((DSQRT(B1(J)**2+B2(J)**2))**3)+B1(J)/((B1(J)**2+B2(J)**2)
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```
2*(2.*B1(J)**2+B2(J)**2)))
 FM08=CC9*U1(J,X,VL,VVP)**2*U1PX(VVP,VL)*(B1(J)*DATAN(B2(J)/DSQRT(
1B1(J)**2+B2(J)**2))/(B2(J)**2*DSQRT(B1(J)**2+B2(J)**2))+DATAN(
2B1(J)/(1.414*B2(J)))/(1.414*B2(J)**2))
 FM09=CC10*U1(J,X,VL,VVP)**3*B2P(J)*(DATAN(B1(J)/DSQRT(2.*B2(J)**2
1))/(2.*1.414*B2(J)**3)+B1(J)/((2.*B2(J)**2)*(B1(J)**2+2.*B2(J)**2
2)))
 FM10=CC11*U1(J,X,VL,VVP)**3*B1P(J)*(DATAN(B2(J)/DSQRT(B1(J)**2+
1B2(J)**2))/((DSQRT(B1(J)**2+B2(J)**2))**3)+B2(J)/((B1(J)**2+B2(J)
2**2)*(B1(J)**2+2.*B2(J)**2)))
 FM11=CC12*U1(J,X,VL,VVP)**2*U1PX(VVP,VL)*(DATAN(0.70700D0)/(B2(J)
1**2))
 FM12=CC13*U1(J,X,VL,VVP)**3*B2P(J)*(DATAN(0.70700D0)/(2.*1.414*
1B2(J)**3)+1./(6.*B2(J)**3))
 FM13=CC14*U1(J,X,VL,VVP)**2*U1PX(VVP,VL)*(DATAN(B1(J)/DSQRT(B1(J)
1**2+B2(J)**2))/((DSQRT(B1(J)**2+B2(J)**2))**3)+B1(J)/(B1(J)**2+
2B2(J)**2))
 FM14=CC15*U1(J,X,VL,VVP)**2*U1PX(VVP,VL)*B1(J)*(DATAN(B2(J)/DSQRT(
1B1(J)**2+B2(J)**2))/((DSQRT(B1(J)**2+B2(J)**2))**3)+B2(J)/((B1(J)
2**2+B2(J)**2)*(B1(J)**2+2.*B2(J)**2)))
 FM15=CC16*B1(J)*B1P(J)/((2.*B1(J)**2+B2(J)**2)**2)
 FM16=CC17*U1(J,X,VL,VVP)**2*U1PX(VVP,VL)/(2.*B1(J)**2+B2(J)**2)
 FM17=CC18*U1(J,X,VL,VVP)**3*B1P(J)/(B1(J)*(2.*B1(J)**2+B2(J)**2))
 FM18=CC19*U1(J,X,VL,VVP)**3*B1(J)*B1P(J)/((2.*B1(J)**2+B2(J)**2)
1**2)
 FM19=CC20*U1(J,X,VL,VVP)**3*B1(J)*B2P(J)/((B1(J)**2+2.*B2(J)**2)
1**2)
 FM2 =CC21*U1(J,X,VL,VVP)**2*B1(J)*U1PX(VVP,VL)/(B2(J)*(B1(J)**2+
12.*B2(J)**2))
 FM21=CC22*U1(J,X,VL,VVP)**3*B1(J)*B2P(J)/(B2(J)**2*(B1(J)**2+
12.*B2(J)**2))
 FM22=CC23*U1(J,X,VL,VVP)**3*B1(J)*B2P(J)/((B1(J)**2+2.*B2(J)**2)
1**2)
 FM23=CC24*U1(J,X,VL,VVP)**2*U1PX(VVP,VL)/(B1(J)**2+B2(J)**2)
 FM24=CC25*U1(J,X,VL,VVP)**3*B1P(J)/(B1(J)*(B1(J)**2+B2(J)**2))
 FM25=CC26*U1(J,X,VL,VVP)**2*U1PX(VVP,VL)*B1(J)/(B2(J)*(B1(J)**2+
1B2(J)**2))
 FM26=CC27*U1(J,X,VL,VVP)**3*B1(J)*B2P(J)/(B2(J)**2*(B1(J)**2+
1B2(J)**2))
 FM27=CC28*U1(J,X,VL,VVP)**2*B2(J)*U1PX(VVP,VL)*(DATAN(B1(J)/(
11.414*B2(J)))/(2.828*B2(J)**3)+B1(J)/(2.*B2(J)**2*(B1(J)**2+2.*
2B2(J)**2)))
 FM28=CC29*U1(J,X,VL,VVP)**2*U1PX(VVP,VL)*B2(J)*(DATAN(0.70700D0)
1/(2.828*B2(J)**3)+1./(6.*B2(J)**3))
 FM29=CC30*U1(J,X,VL,VVP)**3*B2(J)*B1P(J)/((B1(J)**2+2.*B2(J)**2)
1**2)
 FM30=CC31*U1(J,X,VL,VVP)**2*U1PX(VVP,VL)*B2(J)/(B1(J)*(B1(J)**2+
12.*B2(J)**2))
 FM31=CC32*U1(J,X,VL,VVP)**3*B2(J)*B1P(J)/(B2(J)**2*(B1(J)**2+
12.*B2(J)**2))
 FM32=CC33*U1(J,X,VL,VVP)**3*B2(J)*B1P(J)/((B1(J)**2+2.*B2(J)**2)
1**2)
 FM33=CC34*U1(J,X,VL,VVP)**3*B2P(J)/(B2(J)**3)
 FM34=CC35*U1(J,X,VL,VVP)**2*U1PX(VVP,VL)/(B2(J)**2)
 FM35=CC36*U1(J,X,VL,VVP)**3*B2P(J)/(B2(J)**3)
 FM36=CC37*U1(J,X,VL,VVP)**3*B2P(J)/(B2(J)**3)
 FM37=CC38*U1(J,X,VL,VVP)**2*U1PX(VVP,VL)/(B1(J)*B2(J))
 FM38=CC39*U1(J,X,VL,VVP)**3*B1P(J)/(B1(J)**2*B2(J))
 FM39=CC40*U1(J,X,VL,VVP)**2*U1PX(VVP,VL)/(B2(J)**2)
 FM40=CC41*U1(J,X,VL,VVP)**3*B2P(J)/(B2(J)**3)
 FM41=CC42*U1(J,X,VL,VVP)**2*B1(J)*DYB/(B1(J)**2+B2(J)**2)
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 FM42=CC43*U1(J,X,VL,VVP)**2*DYB/B2(J)
 FM43=CC44*U1(J,X,VL,VVP)*DU1(J,X,VL,VVPP,VVP)*R*VTHT1*A1(J)/
1((R*VTHT1+ETA*P+R*VTW)*B2(J)**2*DSQRT(A1(J)**2+B2(J)**2))
 FM44=CC45*U1(J,X,VL,VVP)*DU1(J,X,VL,VVPP,VVP)*R*VTHT1*A2(J)/
1((R*VTHT1+ETA*P+R*VTW)*B2(J)**2*DSQRT(A2(J)**2+B2(J)**2))
 FM45=CC46*U1(J,X,VL,VVP)*DU1(J,X,VL,VVPP,VVP)*(ETA*P+R*VTW)/
1(B2(J)**2*(R*VTHT1+ETA*P+R*VTW))
 FM46=CC47*U1(J,X,VL,VVP)**2*U1PX(VVP,VL)*R*VTHT1*A1(J)/(B2(J)**2
1*(R*VTHT1+ETA*P+R*VTW)*DSQRT(A1(J)**2+B2(J)**2))
 FM47=CC48*U1(J,X,VL,VVP)**2*U1PX(VVP,VL)*R*VTHT1*A2(J)/((R*VTHT1+
1ETA*P+R*VTW)*B2(J)**2*DSQRT(A2(J)**2+B2(J)**2))
 FM48=CC49*U1(J,X,VL,VVP)**2*U1PX(VVP,VL)*(ETA*P+R*VTW)/(B2(J)**2
1*(R*VTHT1+ETA*P+R*VTW))
 FM0001=VTHT1**2*A1(J)**2/((A1(J)**2+B1(J)**2+B2(J)**2)*DSQRT(
12.*A1(J)**2+B1(J)**2+B2(J)**2))
 FM0002=VTHT1**2*A2(J)**2/((A2(J)**2+B1(J)**2+B2(J)**2)*DSQRT(
12.*A2(J)**2+B1(J)**2+B2(J)**2))
 FM0003=VTHT1**2*A1(J)*A2(J)/((A1(J)**2+B1(J)**2+B2(J)**2)*DSQRT(
1A1(J)**2+A2(J)**2+B1(J)**2+B2(J)**2))
 FM0004=VTHT1**2*A2(J)**2/((A2(J)**2+B1(J)**2+B2(J)**2)*DSQRT(
12.*A2(J)**2+B1(J)**2+B2(J)**2))
 FM0005=VTHT1*VTW*A1(J)/(A1(J)**2+B1(J)**2+B2(J)**2)
 FM0006=VTHT1*VTW*A2(J)/(A2(J)**2+B1(J)**2+B2(J)**2)
 FM001=2.*R*C/(DSQRT(PI)*P)*(FM0001+FM0002+FM0003+FM0004+FM0005+
1FM0006)
 Z1=(A1(J)**2*(2.*A1(J)**2+3.*(B1(J)**2+B2(J)**2)))/(4.*(B1(J)**2+
1B2(J)**2)**2*DSQRT(A1(J)**2+B1(J)**2+B2(J)**2)**3)
 Z2=-A1(J)**3/3.*(A1(J)*(15.*(B1(J)**2+B2(J)**2)**2+20.*A1(J)**2*
1(B1(J)**2+B2(J)**2)+8.*A1(J)**4)/(8.*(B1(J)**2+B2(J)**2)**3*DSQRT(
2B1(J)**2+B2(J)**2+A1(J)**2)**5))
 Z3=A1(J)**5/10.*(A1(J)*(105.*(B1(J)**2+B2(J)**2)**3+210.*A1(J)**2*
1(B1(J)**2+B2(J)**2)**2+168.*A1(J)**4*(B1(J)**2+B2(J)**2)+48.*
2A1(J)**6)/(16.*(B1(J)**2+B2(J)**2)**4*DSQRT(A1(J)**2+B1(J)**2+
3B2(J)**2)**7))
 Z4=-A1(J)**7/42.*(A1(J)*(945.*(B1(J)**2+B2(J)**2)**4+2520.*A1(J)
1**2*(B1(J)**2+B2(J)**2)**3+3024.*A1(J)**4*(B1(J)**2+B2(J)**2)**2+
21728.*(B1(J)**2+B2(J)**2)*A1(J)**6+384.*A1(J)**8)/(32.*(B1(J)**2+
3B2(J)**2)**5*DSQRT(A1(J)**2+B1(J)**2+B2(J)**2)**9))
 FM517=1.16*(Z1+Z2+Z3+Z4)
 ZZ1=(A1(J)*A2(J)*(2.*A2(J)**2+3.*(B1(J)**2+B2(J)**2)))/(4.*(B1(J)*
1*2+B2(J)**2)**2*DSQRT(A2(J)**2+B1(J)**2+B2(J)**2)**3)
 ZZ2=-A1(J)**3/3.*(A2(J)*(15.*(B1(J)**2+B2(J)**2)**2+20.*A2(J)**2*(
1B1(J)**2+B2(J)**2)+8.*A2(J)**4)/(8.*(B1(J)**2+B2(J)**2)**3*DSQRT(
2A2(J)**2+B1(J)**2+B2(J)**2)**5))
 ZZ3=A1(J)**5/10.*(A2(J)*(105.*(B1(J)**2+B2(J)**2)**3+210.*A2(J)**2
1*(B1(J)**2+B2(J)**2)**2+168.*A2(J)**4*(B1(J)**2+B2(J)**2)+48.*
2A2(J)**6)/(16.*(B1(J)**2+B2(J)**2)**4*DSQRT(A2(J)**2+B1(J)**2+
3B2(J)**2)**7))
 ZZ4=-A1(J)**7/42.*(A2(J)*(945.*(B1(J)**2+B2(J)**2)**4+2520.*A2(J)
1**2*(B1(J)**2+B2(J)**2)**3+3024.*A2(J)**4*(B1(J)**2+B2(J)**2)**2+
21728.*A2(J)**6*(B1(J)**2+B2(J)**2)+384.*A2(J)**8)/(32.*(B1(J)**2+
3B2(J)**2)**5*DSQRT(A2(J)**2+B1(J)**2+B2(J)**2)**9))
 FM518=1.16*(ZZ1+ZZ2+ZZ3+ZZ4)
 ZZZ1=(A2(J)**2*(2.*A2(J)**2+3.*(B1(J)**2+B2(J)**2)))/(4.*(B1(J)**2
1+B2(J)**2)**2*DSQRT(A2(J)**2+B1(J)**2+B2(J)**2)**3)
 ZZZ2=-A2(J)**3/3.*(A2(J)*(15.*(B1(J)**2+B2(J)**2)**2+20.*A2(J)**2*
1(B1(J)**2+B2(J)**2)+8.*A2(J)**4)/(8.*(B1(J)**2+B2(J)**2)**3*DSQRT(
2A2(J)**2+B1(J)**2+B2(J)**2)**5))
 ZZZ3=A2(J)**5/10.*(A2(J)*(105.*(B1(J)**2+B2(J)**2)**3+210.*A2(J)
1**2*(B1(J)**2+B2(J)**2)**2+168.*A2(J)**4*(B1(J)**2+B2(J)**2)+48.*
2A2(J)**6)/(16.*(B1(J)**2+B2(J)**2)**4*DSQRT(A2(J)**2+B1(J)**2+
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3B2(J)**2)**7))
 ZZZ4=-A2(J)**7/42.*(A2(J)*(945.*(B1(J)**2+B2(J)**2)**4+2520.*A2(J)
1**2*(B1(J)**2+B2(J)**2)**3+3024.*A2(J)**4*(B1(J)**2+B2(J)**2)**2+
21728.*A2(J)**6*(B1(J)**2+B2(J)**2)+384.*A2(J)**8)/(32.*(B1(J)**2+
3B2(J)**2)**5*DSQRT(A2(J)**2+B1(J)**2+B2(J)**2)**9))
 FM519=1.16*(ZZZ1+ZZZ2+ZZZ3+ZZZ4)
 FMO 7=VTHT1**2*FM517+2.*VTHT1**2*FM518+VTHT1**2*FM519
 FFM10=VTHT1*VTW/DSQRT(PI)*(DATAN(A1(J)/DSQRT(B1(J)**2+B2(J)**2))
1/((DSQRT(B1(J)**2+B2(J)**2))**3)+A1(J)/((B1(J)**2+B2(J)**2)*(A1(J)
2**2+B1(J)**2+B2(J)**2)))
 FFM11=VTHT1*VTW/DSQRT(PI)*(DATAN(A2(J)/DSQRT(B1(J)**2+B2(J)**2))
1/((DSQRT(B1(J)**2+B2(J)**2))**3)+A2(J)/((B1(J)**2+B2(J)**2)*(
2A1(J)**2+B1(J)**2+B2(J)**2)))
 FFM12=DSQRT(PI)*VTW**2/(4.*DSQRT(B1(J)**2+B2(J)**2)*(B1(J)**2+
1B2(J)**2))
 FMO08=-2.*R*C*B1(J)**2/P*(FMO07+FFM10+FFM11+FFM12)
 FFM13=C*ETA*VTHT1*A1(J)/(DSQRT(PI)*(A1(J)**2+B1(J)**2+B2(J)**2))
 FFM14=C*ETA*VTHT1*A2(J)/(DSQRT(PI)*(A2(J)**2+B1(J)**2+B2(J)**2))
 FFM15=VTHT1/(2.*DSQRT(PI))*(DATAN(A1(J)/DSQRT(B1(J)**2+B2(J)**2
1))/((DSQRT(B1(J)**2+B2(J)**2))**3)+A1(J)/((B1(J)**2+B2(J)**2)*(
2A1(J)**2+B1(J)**2+B2(J)**2)))
 FFM16=VTHT1/(2.*DSQRT(PI))*(DATAN(A2(J)/DSQRT(B1(J)**2+B2(J)**2
1))/((DSQRT(B1(J)**2+B2(J)**2))**3)+A2(J)/((B1(J)**2+B2(J)**2)*(
2A2(J)**2+B1(J)**2+B2(J)**2)))
 FFM17=VTW*DSQRT(PI)/(4.*DSQRT(B1(J)**2+B2(J)**2)*(B1(J)**2+
1B2(J)**2))
 FMO09=-2.*ETA*C*B1(J)**2*(FFM15+FFM16+FFM17)
 FM501=FMO01+FMO08+FFM13+FFM14+FMO09
 FM49=CC5.*U1(J,X,VL,VVP)**2*B1(J)*FM501
 FFM1=1.414*VTHT1**2*A1(J)**2/(2.*(A1(J)**2+2.*B2(J)**2)*DSQRT(A1(J
1)**2+B2(J)**2))
 FFM2=VTHT1**2*A1(J)*A2(J)/((A2(J)**2+2.*B2(J)**2)*DSQRT(A1(J)**2+
1A2(J)**2+2.*B2(J)**2))
 FFM3=VTHT1**2*A1(J)*A2(J)/((A1(J)**2+2.*B2(J)**2)*DSQRT(A1(J)**2+
1A2(J)**2+2.*B2(J)**2))
 FFM4=1.414*VTHT1**2*A2(J)**2/(2.*(A2(J)**2+2.*B2(J)**2)*DSQRT(
1A2(J)**2+B2(J)**2))
 FFM5=VTHT1*A1(J)*VTW/(A1(J)**2+2.*B2(J)**2)
 FFM6=VTHT1*A2(J)*VTW/(A2(J)**2+2.*B2(J)**2)
 FF1=2.*R*C/(DSQRT(PI)*P)*(FFM1+FFM2+FFM3+FFM4+FFM5+FFM6)
 X1=(A1(J)**2*(2.*A1(J)**2+6.*B2(J)**2))/(16.*B2(J)**4*(DSQRT(2.*
1B2(J)**2+A1(J)**2)**3))
 X2=-A1(J)**3/3.*(A1(J)*(60.*B2(J)**4+40.*A1(J)*B2(J)**2+8.*A1(J)
2**4)/(64.*B2(J)**6*DSQRT(2.*B2(J)**2+A1(J)**2)**5))
 X3=A1(J)**5/10.*(A1(J)*(105.*(2.*B2(J)**2)**3+210.*(2.*B2(J)**2*
1A1(J))**2+168.*(2.*B2(J)**2)*A1(J)**4+48.*A1(J)**6)/(16.*(2.*B2(J)
2**2)**4*DSQRT(2.*B2(J)**2+A1(J)**2)**7))
 X4=-A1(J)**7/42.*(A1(J)*(945.*(2.*B2(J)**2)**4+2520.*(2.*B2(J)**2)
1**3*A1(J)**2+3024.*(2.*B2(J)**2)**2*A1(J)**4+1728.*(2.*B2(J)**2)
2*A1(J)**6+384.*A1(J)**4)/(32.*(2.*B2(J)**2)**5*DSQRT(A1(J)**2+
32.*B2(J)**2)**9))
 FM527=1.16*(X1+X2+X3+X4)
 W1=(A1(J)*A2(J)*(2.*A2(J)**2+6.*B2(J)**2))/(4.*(2.*B2(J)**2)**2*
1DSQRT(2.*B2(J)**2+A2(J)**2)**3)
 W2=-A1(J)**3/3.*(A2(J)*(15.*(2.*B2(J)**2)**2+20.*2.*B2(J)**2*A2(J)
1+8.*A2(J)**4)/(8.*(2.*B2(J)**2)**3*DSQRT(2.*B2(J)**2+A2(J)**2)**5
2))
 W3=A1(J)**5/10.*(A2(J)*(105.*(2.*B2(J)**2)**3+210.*(2.*B2(J)**2*A2
1(J))**2    +168.*2.*B2(J)**2*A2(J)**4+48.*A2(J)**6)/(16.*(2.*B2(J)**
22)**4*DSQRT(2.*B2(J)**2+A2(J)**2)**7))
 W4=-A1(J)**7/42.*(A2(J)*(945.*(2.*B2(J)**2)**4+2520.*(2.*B2(J)**2)
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1**3*A2(J)**2+1/2..*2.*B2(J)**2*A2(J)**6+384.*A2(J)**4+3024.*(2.*
2B2(J)**2)**2*A2(J)**4)/(32.*(2.*B2(J)**2)**5*DSQRT(2.*B2(J)**2+
3A2(J)**2)**3))
 FM528=1.16*(W1+W2+W3+W4)
 WW1=(A2(J)**2*(2.*A2(J)**2+6.*B2(J)**2))/(16.*B2(J)**4*DSQRT(
12.*B2(J)**2+A2(J)**2)**3)
 WW2=-A2(J)**3/3.*(A2(J)*(60.*B2(J)**4+40.*A2(J)*B2(J)**2+8.*A2(J)*
1*4)/(64.*B2(J)**6*DSQRT(2.*B2(J)**2+A2(J)**2)**5))
 WW3=A2(J)**5/1 .*(A2(J)*(1 5.*(2.*B2(J)**2)**3+210.*(2.*B2(J)**2)
1**2*A2(J)**2+168.*2.*B2(J)**2*A2(J)**4+48.*A2(J)**6)/(16.*(2.*
2B2(J)**2)**4*DSQRT(2.*B2(J)**2+A2(J)**2)**7))
 WW4=-A2(J)**7/42.*(A2(J)*(945.*(2.*B2(J)**2)**4+252 .*(2.*B2(J)
1**2)**3*A2(J)**2+3024.*(2.*B2(J)**2)**2*A2(J)**4+1728.*2.*B2(J)**2
2*A2(J)**6+384.*A2(J)**4)/(32.*(2.*B2(J)**2)**5*DSQRT(A2(J)**2+
32.*B2(J)**2)**9))
 FM529=1.16*(WW1+WW2+WW3+WW4)
 FFM7=VTHT1**2*FM527+2.*VTHT1**2*FM528+VTHT1**2*FM529
 FFM10=VTHT1*VTW/DSQRT(PI)*(DATAN(A1(J)/(1.414*B2(J)))/(2.828*B2(J)
1**3)+A1(J)/(2.*B2(J)**2*(B1(J)**2+2.*B2(J)**2)))
 FFM11=VTHT1*VTW/DSQRT(PI)*(DATAN(A2(J)/(1.414*B2(J)))/(2.828*B2(J)
1**3)+A2(J)/(2.*B2(J)**2*(B1(J)**2+2.*B2(J)**2)))
 FFM12=VTW**2*DSQRT(PI)/(16.*B2(J)**3)
 FF2=-2.*R*C*B2(J)**2/PX(FFM7+FFM10+FFM11+FFM12)
 FFM13=C*ETA*VTHT1*A1(J)/(DSQRT(PI)*(A1(J)**2+2.*B2(J)**2))
 FFM14=C*ETA*VTHT1*A2(J)/(DSQRT(PI)*(A2(J)**2+2.*B2(J)**2))
 FFM15=VTHT1/(2.*DSQRT(PI))*(DATAN(A1(J)/(1.414*B2(J)))/(2.828*B2(J
1)**3)+A1(J)/((2.*B2(J)**2)*(B1(J)**2+2.*B2(J)**2)))
 FFM16=VTHT1/(2.*DSQRT(PI))*(DATAN(A2(J)/(1.414*B2(J)))/(2.828*B2(J
1)**3)+A2(J)/((2.*B2(J)**2)*(B1(J)**2+2.*B2(J)**2)))
 FFM17=VTW*DSQRT(2.*PI)/(16.*B2(J)**3)
 FF3=-2.*C*ETA*B2(J)**2*(FFM15+FFM16+FFM17)
 FM5 2=FF1+FF2+FF3+FFM13+FFM14
 FM5 =CC51*U1(J,X,VL,VVP)**2*B2(J)*FM5 2
 SUM1=FM01+FM 3+FM 4+FM05+FM 6+FM07+FM 8+FM09+FM10+FM11+FM12+FM13
 SUM2=FM14+FM15+FM16+FM17+FM18+FM19+FM2 +FM21+FM22+FM23+FM24+FM25
 SUM3=FM26+FM27+FM28+FM29+FM30+FM31+FM32+FM33+FM34+FM35+FM36+FM37
 SUM4=FM38+FM39+FM4 +FM41+FM42+FM43+FM44+FM45+FM46+FM47+FM48+FM49
 FM2=(SUM1+SUM2+SUM3+SUM4+FM5 )
 RETURN
 END
 FUNCTION FE1(J,X,A1,A2,B1,B2,A1P,A2P,B1P,B2P,P,VTW,VT1,VTHT1,VL,
1VVP,VVPP,DTHT1,DTW,DT1,RHO1,DRHO1,VT1PX)
 IMPLICIT REAL*8(A-H),REAL*8(O-Z)
 COMMON OMEGA1,OMEGA2,R,PI,ETA,C,PR,DYB,P1,P2,P3,TW1,TW2
 COMMON TW3,T11,T12,T13,THT11,THT12,THT13,RHO0,RHO11,RHO12,RHO13
 COMMON DRHO11,DRHO12,DRHO13,VTWPX,VTHPX,DTW1,DTW2,DTW3,VL1,VL2
 COMMON VL3,OOMGA,VVP1,VVP2,VVP3,VVPP1,VVPP2,VVPP3,H,DT11,DT12,DT13
 COMMON C4,CC51,C49,CC2,CE5,CE3,CCE3,CCE5,CC5
 COMMON T1PX1,T1PX2,T1PX3,DELTA1,DELTA2,J1,CP
 COMMON DTHT11,DTHT12,DTHT13
 DIMENSION X(12),A1(12),A2(12),B1(12),B2(12),A1P(12),A2P(12)
 DIMENSION B1P(12),B2P(12)
 CE1=-.5*CP*DSQRT(2./PI)*DELTA1**2
 CE2=-CP/DSQRT(PI)*DELTA1*DELTA2
 CE3=.25*CP*DSQRT(2./PI)*DELTA1**2
 CE4=-CP/PI
 CE5=CE2
 CE6=-2.*CP/PI
 CE7=-CP/DSQRT(PI)*DELTA1
 CE8=-2.*CP/(PI*DSQRT(PI))*OMEGA1*DELTA1**2
 CE9=-2.*CP/(PI*DSQRT(PI))*OMEGA1*DELTA1*DELTA2
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CE1 =-2.*CP/(PI*DSQRT(PI))*OMEGA1*DELTA1*2
CE11=CE9
CE12=-2.*CP/(PI*DSQRT(PI))*OMEGA2*DELTA1**2
CE13=CE9
CE14=CE12
CE15=-2.*CP/(PI*DSQRT(PI))*OMEGA2*DELTA1*DELTA2
CE16=-CP/DSQRT(PI)*OMEGA1*DELTA1
CE17=-CP/DSQRT(PI)*OMEGA2*DELTA1
CE18=-CE8
CE19=2.*CP/(PI*DSQRT(PI))*OMEGA1*DELTA1*DELTA2
CE2 =2.*CP/(PI*DSQRT(PI))*OMEGA2*DELTA1**2
CE21=2.*CP/(PI*DSQRT(PI))*OMEGA2*DELTA1*DELTA2
CE22=4.*CP/(PI*DSQRT(PI))*OMEGA1*DELTA1**2
CE23=4.*CP/(PI*DSQRT(PI))*OMEGA1*DELTA1*DELTA2
CE24=2.*CP/(PI*DSQRT(PI))*OMEGA1*DELTA1**2
CE25=2.*CP/(PI*DSQRT(PI))*OMEGA1*DELTA1*DELTA2
CE26=CE8
CE27=CE9
CE28=-CE22
CE29=-4.*CP/(PI*DSQRT(PI))*OMEGA1*DELTA1*DELTA2
CE3 =4.*CP/(PI*DSQRT(PI))*OMEGA2*DELTA1**2
CE31=4.*CP/(PI*DSQRT(PI))*OMEGA2*DELTA1*DELTA2
CE32=2.*CP/(PI*DSQRT(PI))*OMEGA2*DELTA1**2
CE33=2.*CP/(PI*DSQRT(PI))*OMEGA2*DELTA1*DELTA2
CE34=-CE32
CE35=-CE33
CE36=CE28
CE37=-4.*CP/(PI*DSQRT(PI))*OMEGA2*DELTA1*DELTA2
CE38=.5*CE8
CE39=CE9
CE40=-CE38
CE41=CE19
CE42=-  CP/(PI*DSQRT(PI))*OMEGA2*DELTA1**2
CE43=-2.*CP/(PI*DSQRT(PI))*OMEGA2*DELTA1*DELTA2
CE44=-CE42
CE45=-CE43
CE46=-CE4
CE47=2.*CE46
CE48=-2.*CE3*C
CE49=2.*CE2*C
CE50=4.*CP/(PI*DSQRT(PI))*OMEGA1**2*DELTA1
CE51=8.*CP/(PI*DSQRT(PI))*OMEGA1*OMEGA2*DELTA1
CE52=4.*CP/(PI*DSQRT(PI))*OMEGA2**2*DELTA1
CE53=-CE1
CE54=-CE2
CE55=CE54/DELTA2
CE56=CE53
CE57=CE54
CE58=CE54/DELTA2
CE59=CE8/CP
CE60=-2.*OMEGA1*DELTA1*DELTA2/(PI*DSQRT(PI))
CE61=-OMEGA1*DELTA1/DSQRT(PI)
CE62=-2.*OMEGA2*DELTA1**2/(PI*DSQRT(PI))
CE63=-2.*OMEGA2*DELTA1*DELTA2/(PI*DSQRT(PI))
CE64=-OMEGA2*DELTA1/DSQRT(PI)
CE65=-2.*OMEGA1*DELTA1**2/(PI*DSQRT(PI))
CE66=-2.*OMEGA1*DELTA1*DELTA2/(PI*DSQRT(PI))
CE67=CE61
CE68=CE62
CE69=-2.*OMEGA2*DELTA1*DELTA2/(PI*DSQRT(PI))
CE70=-OMEGA2*DELTA1/DSQRT(PI)
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      CE71=.5*DSQRT(2./PI)*DELTA1**2
      CE72=DELTA1*DELTA2/DSQRT(PI)
      CE73=DELTA1/DSQRT(PI)
      CE74=.5*DSQRT(2./PI)*DELTA1**2
      CE75=CE72
      CE76=DELTA1/DSQRT(PI)
      FFE1=CE1*VTHT1*DTHT1/(A1(J)**2)
      FFE2=CE2*VTHT1*DTHT1*A2(J)/(A1(J)**2*DSQRT(A1(J)**2+A2(J)**2))
      FFE3=CE4*VTHT1**2*DYB/A1(J)
      FFE5=CE6*VTHT1**2*A2(J)*DYB/(A1(J)**2+A2(J)**2)+CE7*VTHT1*DTW/
     1(A1(J)**2)
      FFE6=CE8*VTHT1*VTHPX*U1(J,X,VL,VVP)*(DATAN(B1(J)/(1.414*A1(J)))/
     1(1.414*A1(J))+B1(J)*DATAN(A1(J)/DSQRT(A1(J)**2+B1(J)**2))/
     2A1(J)**2*DSQRT(A1(J)**2+B1(J)**2))
      FFE7=CE9*VTHT1*VTHPX*U1(J,X,VL,VVP)*(A2(J)*DATAN(B1(J)/DSQRT(
     1A1(J)**2+A2(J)**2))/(A1(J)**2*DSQRT(A1(J)**2+A2(J)**2))+B1(J)*
     2DATAN(A2(J)/DSQRT(A1(J)**2+B1(J)**2))/(A1(J)**2*DSQRT(A1(J)**2+
     3B1(J)**2)))
      FFE8=CE10*VTHT1**2*U1(J,X,VL,VVP)*A1P(J)*(DATAN(B1(J)/(1.414*A1(J)
     1))/(2.828*A1(J)**3)+B1(J)/(2.*A1(J)**2*(B1(J)**2+2.*A1(J)**2)))
      FFE9=CE11*VTHT1**2*U1(J,X,VL,VVP)*A2P(J)*(DATAN(B1(J)/DSQRT(
     1A1(J)**2+A2(J)**2))/((DSQRT(A1(J)**2+A2(J)**2))**3)+B1(J)/((
     2A1(J)**2+A2(J)**2)*(A1(J)**2+A2(J)**2+B1(J)**2)))
      FFE10=CE12*VTHT1*VTHPX*U1(J,X,VL,VVP)*(DATAN(B2(J)/(1.414*A1(J)))
     1/(1.414*A1(J)**2)+B2(J)*DATAN(A1(J)/DSQRT(A1(J)**2+B2(J)**2))/(
     2A1(J)**2*DSQRT(A1(J)**2+B2(J)**2)))
      FFE11=CE13*VTHT1*VTHPX*U1(J,X,VL,VVP)*(A2(J)*DATAN(B2(J)/DSQRT(
     1A1(J)**2+A2(J)**2))/(A1(J)**2*DSQRT(A1(J)**2+A2(J)**2))+B2(J)*
     2DATAN(A2(J)/DSQRT(A1(J)**2+B2(J)**2))/(A1(J)**2*DSQRT(A1(J)**2+
     3B2(J)**2)))
      FFE12=CE14*VTHT1**2*U1(J,X,VL,VVP)*A1P(J)*(DATAN(B2(J)/(1.414*
     1A1(J)))/(2.828*A1(J)**3)+B2(J)/(2.*A1(J)**2*(B2(J)**2+2.*A1(J)**2
     2)))
      FFE13=CE15*VTHT1**2*U1(J,X,VL,VVP)*A2P(J)*(DATAN(B2(J)/DSQRT(
     1A1(J)**2+A2(J)**2))/((DSQRT(A1(J)**2+A2(J)**2))**3)+B2(J)/((
     2A1(J)**2+A2(J)**2)*(A1(J)**2+A2(J)**2+B2(J)**2)))
      FFE14=CE16*VTHT1*U1(J,X,VL,VVP)*VTWPX*B1(J)/(A1(J)**2*DSQRT(
     1A1(J)**2+B1(J)**2))
      FFE15=CE17*VTHT1*U1(J,X,VL,VVP)*VTWPX*B2(J)/(A1(J)**2*DSQRT(
     1A1(J)**2+B2(J)**2))
      FFE16=CE18*VTHT1**2*U1PX(VVP,VL)*A1(J)*(DATAN(B1(J)/(1.414*A1(J)
     1))/(2.828*A1(J)**3)+B1(J)/(2.*A1(J)**2*(B1(J)**2+2.*A1(J)**2)))
      FFE17=CE19*VTHT1**2*U1PX(VVP,VL)*A2(J)*(DATAN(B1(J)/DSQRT(A1(J)**2
     1+A2(J)**2))/((DSQRT(A1(J)**2+A2(J)**2))**3)+B1(J)/((A1(J)**2+
     2A2(J)**2)*(A1(J)**2+A2(J)**2+B1(J)**2)))
      FFE18=CE20*VTHT1**2*U1PX(VVP,VL)*A1(J)*(DATAN(B2(J)/(1.414*A1(J)
     1))/(2.828*A1(J)**3)+B2(J)/(2.*A1(J)**2*(B2(J)**2+2.*A1(J)**2)))
      FFE19=CE21*VTHT1**2*U1PX(VVP,VL)*A2(J)*(DATAN(B2(J)/DSQRT(A1(J)**2
     1+A2(J)**2))/((DSQRT(A1(J)**2+A2(J)**2))**3)+B2(J)/((A1(J)**2+
     2A2(J)**2)*(A1(J)**2+A2(J)**2+B2(J)**2)))
      FFE20=CE22*VTHT1**2*U1PX(VVP,VL)*B1P(J)*A1(J)/((2.*A1(J)**2+
     1B1(J)**2)**2)
      FFE21=CE23*VTHT1**2*U1(J,X,VL,VVP)*B1P(J)*A2(J)/((A1(J)**2+
     1A2(J)**2+B1(J)**2)**2)
      FFE22=CE24*VTHT1**2*U1PX(VVP,VL)*A1(J)/(B1(J)*(2.*A1(J)**2+B1(J)**
     12))
      FFE23=CE25*VTHT1**2*U1PX(VVP,VL)*A2(J)/(B1(J)*DSQRT(A1(J)**2+
     1A2(J)**2+B1(J)**2))
      FFE24=CE26*VTHT1**2*U1(J,X,VL,VVP)*B1P(J)*A1(J)/(B1(J)**2*(
     12.*A1(J)**2+B1(J)**2))
      FFE25=CE27*VTHT1**2*U1(J,X,VL,VVP)*B1P(J)*A2(J)/(B1(J)**2*(
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1A1(J)**2+A2(J)**2+B1(J)**2))
 FFE26=CE28*VTHT1**2*U1(J,X,VL,VVP)*B1P(J)*A1(J)/((2.*A1(J)**2+
1B1(J)**2)**2)
 FFE27=CE29*VTHT1**2*U1(J,X,VL,VVP)*B1P(J)*A2(J)/((A1(J)**2+A2(J)
1**2+B1(J)**2)**2)
 FFE28=CE30*VTHT1**2*U1(J,X,VL,VVP)*B2P(J)*A1(J)/((2.*A1(J)**2+
1B2(J)**2)**2)
 FFE29=CE31*VTHT1**2*U1(J,X,VL,VVP)*B2P(J)*A2(J)/((A1(J)**2+
1A2(J)**2+B2(J)**2)**2)
 FFE30=CE32*VTHT1**2*U1PX(VVP,VL)*A1(J)/(B2(J)*(2.*A1(J)**2+
1B2(J)**2))
 FFE31=CE33*VTHT1**2*U1PX(VVP,VL)*A2(J)/(B2(J)*(A1(J)**2+
1A2(J)**2+B2(J)**2))
 FFE32=CE34*VTHT1**2*U1(J,X,VL,VVP)*B2P(J)*A1(J)/(B2(J)**2*(
12.*A1(J)**2+B2(J)**2))
 FFE33=CE35*VTHT1**2*U1(J,X,VL,VVP)*B2P(J)*A2(J)/(B2(J)**2*(
1A1(J)**2+A2(J)**2+B2(J)**2))
 FFE34=CE36*VTHT1**2*U1(J,X,VL,VVP)*B2P(J)*A1(J)/((2.*A1(J)**2+
1B2(J)**2)**2)
 FFE35=CE37*VTHT1**2*U1(J,X,VL,VVP)*B2P(J)*A2(J)/((A1(J)**2+
1A2(J)**2+B2(J)**2)**2)
 FFE36=CE38*VTHT1**2*U1PX(VVP,VL)/(A1(J)*B1(J))
 FFE37=CE39*VTHT1**2*U1PX(VVP,VL)*A2(J)/(B1(J)*(A1(J)**2+A2(J)**2))
 FFE38=CE40*VTHT1**2*U1(J,X,VL,VVP)*B1P(J)/(A1(J)*B1(J)**2)
 FFE39=CE41*VTHT1**2*U1(J,X,VL,VVP)*B1P(J)*A2(J)/(B1(J)**2*(
1A1(J)**2+A2(J)**2))
 FFE40=CE42*VTHT1**2*U1PX(VVP,VL)/(A1(J)*B2(J))
 FFE41=CE43*VTHT1**2*U1PX(VVP,VL)*A2(J)/(B2(J)*(A1(J)**2+A2(J)**2))
 FFE42=CE44*VTHT1**2*U1(J,X,VL,VVP)*B2P(J)/(A1(J)*B2(J)**2)
 FFE43=CE45*VTHT1**2*U1(J,X,VL,VVP)*B2P(J)*A2(J)/(B2(J)**2*(
1A1(J)**2+A2(J)**2))
 FFE44=CE46*VTHT1**2*DYB/A1(J)
 FFE45=CE47*VTHT1**2*DYB*A2(J)/(A1(J)**2+A2(J)**2)
 FFE46=CE48*VTHT1**2*RHO1*VT1/(PR*RHO0**2)
 FFE47=CE49*VTHT1**2*A2(J)**3*RHO1*VT1/(RHO0**2*PR*(A1(J)**2+
1A2(J)**2)*DSQRT(A1(J)**2+A2(J)**2))
 FFE48=CE50*VTHT1*U1(J,X,VL,VVP)**2*B1(J)**2*VT1*RHO1/
1(RHO0**2*(A1(J)**2+2.*B1(J)**2))
 FFE49=CE51*VTHT1*U1(J,X,VL,VVP)**2*B1(J)*B2(J)*VT1*RHO1/
1(RHO0**2*(A1(J)**2+B1(J)**2+B2(J)**2))
 FFE50=CE52*VTHT1*U1(J,X,VL,VVP)**2*B2(J)**2*VT1*RHO1/
1(RHO0**2*(A1(J)**2+2.*B2(J)**2))
 FFE51=CE53*VTHT1**2*DT1*R/(R*VTHT1+ETA*P+R*VTW)
 FFE52=CE54*VTHT1**2*DT1*R*A2(J)/((R*VTHT1+ETA*P+R*VTW)*
1A1(J)**2*DSQRT(A1(J)**2+A2(J)**2))
 FFE53=CE55*VTHT1*DT1*(ETA*P+R*VTW)/(A1(J)**2*(R*VTHT1+ETA*P+R*VTW
1))
 FFE54=CE56*VTHT1**2*U1(J,X,VL,VVP)*VT1PX*R/(A1(J)**2*(
1R*VTHT1+ETA*P+R*VTW))
 FFE55=CE57*VTHT1**2*U1(J,X,VL,VVP)*VT1PX*R*A2(J)/(A1(J)**2
1*DSQRT(A1(J)**2+A2(J)**2)*(R*VTHT1+ETA*P+R*VTW))
 FFE56=CE58*VTHT1*U1(J,X,VL,VVP)*VT1PX*(ETA*P+R*VTW)/(A1(J)**2*
1(R*VTHT1+ETA*P+R*VTW))
 FFE57=CE59*VTHT1**2*U1(J,X,VL,VVP)*DU1(J,X,VL,VVPP,VVP)*R*(DATAN(
1B1(J)/(1.414*A1(J)))/(1.414)+B1(J)*DATAN(A1(J)/DSQRT(A1(J)**2+
2B1(J)**2))/DSQRT(A1(J)**2+B1(J)**2))/(A1(J)**2*(R*VTHT1+ETA*P+
3R*VTW))
 FFE58=CE60*VTHT1**2*U1(J,X,VL,VVP)*DU1(J,X,VL,VVPP,VVP)*R*(DATAN(
1B1(J)/(1.414*A2(J)))/1.414+B1(J)*DATAN(A2(J)/DSQRT(A2(J)**2+B1(J)
2**2))/DSQRT(A2(J)**2+B1(J)**2))/(A2(J)**2*(R*VTHT1+ETA*P+R*VTW))
 FFE59=CE61*VTHT1*U1(J,X,VL,VVP)*DU1(J,X,VL,VVPP,VVP)*B1(J)*(ETA*P+
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1R*VTW)/(A1(J)**2*DSQRT(A1(J)**2+B1(J)**2)*(R*VTHT1+ETA*P+R*VTW))
 FFE60=CE62*VTHT1**2*U1(J,X,VL,VVP)*DU1(J,X,VL,VVPP,VVP)*R*(DATAN(
1B2(J)/(1.414*A1(J)))/1.414+B2(J)*DATAN(A1(J)/DSQRT(A1(J)**2+B2(J)
2**2))/DSQRT(A1(J)**2+B2(J)**2))/(A1(J)**2*(R*VTHT1+ETA*P+R*VTW))
 FFE61=CE63*VTHT1**2*U1(J,X,VL,VVP)*DU1(J,X,VL,VVPP,VVP)*R*(DATAN(
1B2(J)/(1.414*A2(J)))/1.414+B2(J)*DATAN(A2(J)/DSQRT(A2(J)**2+B2(J)
2**2))/DSQRT(A2(J)**2+B2(J)**2))/(A2(J)**2*(R*VTHT1+ETA*P+R*VTW))
 FFE62=CE64*VTHT1*U1(J,X,VP,VVL)
 FFE62=CE64*VTHT1*U1(J,X,VL,VVP)*DU1(J,X,VL,VVPP,VVP)*B2(J)*(ETA*P+
1R*VTW)/(A1(J)**2*DSQRT(A1(J)**2+B2(J)**2)*(R*VTHT1+ETA*P+R*VTW))
 FFE63=CE65*VTHT1**2*U1(J,X,VL,VVP)**2*U1PX(VVP,VL)*R*(DATAN(B1(J)
1/(1.414*A1(J)))/1.414+B1(J)*DATAN(A1(J)/DSQRT(A1(J)**2+B1(J)**2))
2/DSQRT(A1(J)**2+B1(J)**2))/(A1(J)**2*(R*VTHT1+ETA*P+R*VTW))
 FFE64=CE66*VTHT1**2*U1(J,X,VL,VVP)**2*U1PX(VVP,VL)*R*(DATAN(B1(J)
1/(1.414*A2(J)))/1.414+B1(J)*DATAN(A2(J)/DSQRT(A2(J)**2+B1(J)**2))
2/DSQRT(A2(J)**2+B1(J)**2))/(A2(J)**2*(R*VTHT1+ETA*P+R*VTW))
 FFE65=CE67*VTHT1*U1(J,X,VL,VVP)**2*U1PX(VVP,VL)*B1(J)*(ETA*P+
1R*VTW)/(A1(J)**2*DSQRT(A1(J)**2+B1(J)**2)*(R*VTHT1+ETA*P+R*VTW))
 FFE66=CE68*VTHT1**2*U1(J,X,VL,VVP)**2*U1PX(VVP,VL)*R*(DATAN(B2(J)
1/(1.414*A1(J)))/1.414+B2(J)*DATAN(A1(J)/DSQRT(A1(J)**2+B2(J)**2))/
2DSQRT(A1(J)**2+B2(J)**2))/(A1(J)**2*(R*VTHT1+ETA*P+R*VTW))
 FFE67=CE69*VTHT1**2*U1(J,X,VL,VVP)**2*U1PX(VVP,VL)*R*(DATAN(B2(J)
1/(1.414*A2(J)))/1.414+B2(J)*DATAN(A2(J)/DSQRT(A2(J)**2+B2(J)**2))/
2DSQRT(A2(J)**2+B2(J)**2))/(A2(J)**2*(R*VTHT1+ETA*P+R*VTW))
 FFE68=CE70*VTHT1*U1(J,X,VL,VVP)**2*U1PX(VVP,VL)*B2(J)*(ETA*P+R*VTW
1)/(A1(J)**2*DSQRT(A1(J)**2+B2(J)**2)*(R*VTHT1+ETA*P+R*VTW))
 FFE69=CE71*VTHT1**2*U1(J,X,VL,VVP)*DU1(J,X,VL,VVPP,VVP)*R/(A1(J)**
12*(R*VTHT1+ETA*P+R*VTW))
 FFE70=CE72*VTHT1**2*U1(J,X,VL,VVP)*DU1(J,X,VL,VVPP,VVP)*R*A2(J)/
1(A1(J)**2*DSQRT(A1(J)**2+A2(J)**2)*(R*VTHT1+ETA*P+R*VTW))
 FFE71=CE73*VTHT1*U1(J,X,VL,VVP)*DU1(J,X,VL,VVPP,VVP)*(ETA*P+R*VTW)
1/(A1(J)**2*(R*VTHT1+ETA*P+R*VTW))
 FFE72=CE74*VTHT1**2*U1(J,X,VL,VVP)**2*U1PX(VVP,VL)*R/(A1(J)**2
1*(R*VTHT1+ETA*P+R*VTW))
 FFE73=CE75*VTHT1**2*U1(J,X,VL,VVP)**2*U1PX(VVP,VL)*A2(J)*R/
1(A1(J)**2*DSQRT(A1(J)**2+A2(J)**2)*(R*VTHT1+ETA*P+R*VTW))
 FFE74=CE76*VTHT1*U1(J,X,VL,VVP)**2*U1PX(VVP,VL)*(ETA*P+R*VTW)/(
1A1(J)**2*(R*VTHT1+ETA*P+R*VTW))
 SUM1=FFE1+FFE2+FFE3+FFE5+FFE6+FFE7+FFE8+FFE9+FFE10+FFE11
 SUM2=FFE12+FFE13+FFE14+FFE15+FFE16+FFE17+FFE18+FFE19+FFE20+FFE21
 SUM3=FFE22+FFE23+FFE24+FFE25+FFE26+FFE27+FFE28+FFE29+FFE30+FFE31
 SUM4=FFE32+FFE33+FFE34+FFE35+FFE36+FFE37+FFE38+FFE39+FFE40+FFE41
 SUM5=FFE42+FFE43+FFE44+FFE45+FFE46+FFE47+FFE48+FFE49+FFE50+FFE51
 SUM6=FFE52+FFE53+FFE54+FFE55+FFE56+FFE57+FFE58+FFE59+FFE60+FFE61
 SUM7=FFE62+FFE63+FFE64+FFE65+FFE66+FFE67+FFE68+FFE69+FFE70+FFE71
 SUM8=FFE72+FFE73+FFE74
 FE1=(SUM1+SUM2+SUM3+SUM4+SUM5+SUM6+SUM7+SUM8)
 RETURN
 END
 FUNCTION FE2(J,X,A1,A2,B1,B2,A1P,A2P,B1P,B2P,P,VTW,VT1,VTHT1,VL,
1VVP,VVPP,DTHT1,DTW,DT1,RHO1,DRHO1,VT1PX)
 IMPLICIT REAL*8(A-H),REAL*8(O-Z)
 COMMON OMEGA1,OMEGA2,R,PI,ETA,C,PR,CY0,P1,P2,P3,TW1,TW2
 COMMON TW3,T11,T12,T13,THT11,THT12,THT13,RHO0,RHO11,RHO12,RHO13
 COMMON DRHO11,DRHO12,DRHO13,VTWPX,VTHPX,DTW1,DTW2,DTW3,VL1,VL2
 COMMON VL3,DOMGA,VVP1,VVP2,VVP3,VVPP1,VVPP2,VVPP3,H,DT11,DT12,DT13
 COMMON C4,CC51,C49,CC2,CE5,CE3,CCE3,CCE5,CC5
 COMMON T1PX1,T1PX2,T1PX3,DELTA1,DELTA2,J1,CP
 COMMON DTHT11,DTHT12,DTHT13
 DIMENSION X(12),A1(12),A2(12),B1(12),B2(12),A1P(12),A2P(12)
 DIMENSION B1P(12),B2P(12)
```

```
CCE1=-CP/DSQRT(PI)*DELTA1*DELTA2
CCE2=-.5*DSQRT(2./PI)*CP*DELTA2**2
CCE3=CCE1
CCE4=-CP/PI
CCE5=.25*CP*DSQRT(2./PI)*DELTA2**2
CCE6=CCE4
CCE7=-CP/DSQRT(PI)*DELTA2
CCE8=-2.*CP/(PI*DSQRT(PI))*OMEGA1*DELTA1*DELTA2
CCE9=-2.*CP/(PI*DSQRT(PI))*OMEGA1*DELTA2**2
CCE10=CCE8
CCE11=-2.*CP/(PI*DSQRT(PI))*OMEGA1*DELTA2**2
CCE12=-2.*CP/(PI*DSQRT(PI))*OMEGA2*DELTA1*DELTA2
CCE13=-2.*CP/(PI*DSQRT(PI))*OMEGA2*DELTA2**2
CCE14=CCE12
CCE15=CCE13
CCE16=-CP/DSQRT(PI)*OMEGA1*DELTA2
CCE17=-CP/DSQRT(PI)*OMEGA2*DELTA2
CCE18=-CCE8
CCE19=-CCE9
CCE20=-CCE12
CCE21=-CCE15
CCE22=2.*CCE18
CCE23=2.*CCE19
CCE24=CCE18
CCE25=CCE19
CCE26=CCE8
CCE27=CCE9
CCE28=-CCE22
CCE29=2.*CCE27
CCE30=-2.*CCE14
CCE31=-2.*CCE15
CCE32=CCE20
CCE33=CCE21
CCE34=-CCE20
CCE35=CCE13
CCE36=-CCE30
CCE37=-CCE31
CCE38=CCE8
CCE39=.5*CCE9
CCE40=CCE18
CCE41=-CCE39
CCE42=CCE12
CCE43=-CP/(PI*DSQRT(PI))*OMEGA2*DELTA2**2
CCE44=CCE20
CCE45=-CCE43
CCE46=-2.*CCE4
CCE47=-CCE4
CCE48=2.*CCE1
CCE49=CCE2
CCE50=4.*C*OMEGA1**2*DELTA2/(PI*DSQRT(PI))
CCE51=8./(PI*DSQRT(PI))*OMEGA1*OMEGA2*DELTA2*C
CCE52=4.*C*OMEGA2**2*DELTA2/(PI*DSQRT(PI))
CCE53=-CCE1
CCE54=-CCE2
CCE55=CP*DELTA2/DSQRT(PI)
CCE56=CCE53
CCE57=CCE54
CCE58=CCE55
CCE59=CCE8/CP
CCE60=2.*CCE39
CCE61=CCE16/CP
```

```
      CCE62=CCE42/CP
      CCE63=CCE35/CP
      CCE64=-OMEGA2*DELTA2/DSQRT(PI)
      CCE65=CCE59
      CCE66=-2.*OMEGA1*DELTA2**2/(PI*DSQRT(PI))
      CCE67=CCE61
      CCE68=CCE62
      CCE69=CCE63
      CCE70=CCE64
      CCE71=DELTA1*DELTA2/DSQRT(PI)
      CCE72=.5*DSQRT(2./PI)*DELTA2**2
      CCE73=CCE71/DELTA1
      CCE74=CCE71
      CCE75=CCE72
      CCE76=CCE71/DELTA1
      FFE1=CCE1*VTHT1*DTHT1*A1(J)/(A2(J)**2*DSQRT(A1(J)**2+A2(J)**2))
      FFE2=CCE2*VTHT1*DTHT1/(A2(J)**2)
      FFE4=CCE4*VTHT1**2*A1(J)*DYB/(A1(J)**2+A2(J)**2)
      FFE5=CCE6*VTHT1**2*DYB/A2(J)+CCE7*VTHT1*DTW/(A2(J)**2)
      FFE6=CCE8*VTHT1*VTHPX*U1(J,X,VL,VVP)*(A1(J)*DATAN(B1(J)/DSQRT(
     1A1(J)**2+A2(J)**2))/(A2(J)**2*DSQRT(A1(J)**2+A2(J)**2))+B1(J)*
     2DATAN(A1(J)/DSQRT(A2(J)**2+B1(J)**2))/(A2(J)**2*DSQRT(A2(J)**2+
     3B1(J)**2)))
      FFE7=CCE9*VTHT1*VTHPX*U1(J,X,VL,VVP)*(DATAN(B1(J)/(1.414*A2(J)))/
     1(1.414*A2(J)**2)+B1(J)*DATAN(A2(J)/DSQRT(A2(J)**2+B1(J)**2))/
     2(A2(J)**2*DSQRT(A2(J)**2+B1(J)**2)))
      FFE8=CCE10*VTHT1**2*U1(J,X,VL,VVP)*A1P(J)*(DATAN(B1(J)/DSQRT(A1(J)
     1**2+A2(J)**2))/((DSQRT(A1(J)**2+A2(J)**2))**3)+B1(J)/((A1(J)**2+
     2A2(J)**2)*(A1(J)**2+B1(J)**2)))
      FFE9=CCE11*VTHT1**2*U1(J,X,VL,VVP)*A2P(J)*(DATAN(B1(J)/(1.414*
     1A2(J)))/(2.828*A2(J)**3)+B1(J)/((2.*A2(J)**2)*(2.*A2(J)**2+
     2B1(J)**2)))
      FFE10=CCE12*VTHT1*VTHPX*U1(J,X,VL,VVP)*(A1(J)*DATAN(B2(J)/DSQRT(
     1A1(J)**2+A2(J)**2))/(A2(J)**2*DSQRT(A1(J)**2+A2(J)**2))+B2(J)*
     2DATAN(A1(J)/DSQRT(A2(J)**2+B2(J)**2))/(A2(J)**2*DSQRT(A2(J)**2+B2
     3(J)**2)))
      FFE11=CCE13*VTHT1*VTHPX*U1(J,X,VL,VVP)*(DATAN(B2(J)/(1.414*A2(J)
     1))/(1.414*A2(J)**2)+B2(J)*DATAN(A2(J)/DSQRT(A2(J)**2+B2(J)**2))
     2/(A2(J)**2*DSQRT(A2(J)**2+B2(J)**2)))
      FFE12=CCE14*VTHT1**2*U1(J,X,VL,VVP)*A1P(J)*(DATAN(B2(J)/DSQRT(
     1A1(J)**2+A2(J)**2))/(DSQRT(A1(J)**2+A2(J)**2)**3)+B2(J)/((A1(J)
     2**2+A2(J)**2)*(A1(J)**2+A2(J)**2+B2(J)**2)))
      FFE13=CCE15*VTHT1**2*U1(J,X,VL,VVP)*A2P(J)*(DATAN(B2(J)/(1.414*
     1A2(J)))/(2.828*A2(J)**3)+B2(J)/((2.*A2(J)**2)*(2.*A2(J)**2
     2+B2(J)**2)))
      FFE14=CCE16*VTHT1*U1(J,X,VL,VVP)*VTWPX*B1(J)/(2.*A2(J)**2*
     1DSQRT(A2(J)**2+B1(J)**2))
      FFE15=CCE17*VTHT1*U1(J,X,VL,VVP)*VTWPX*B2(J)/(2.*A2(J)**2*
     1DSQRT(A2(J)**2+B2(J)**2))
      FFE16=CCE18*VTHT1**2*U1PX(VVP,VL)*A1(J)*(DATAN(B1(J)/DSQRT(A1(J)
     1**2+A2(J)**2))/((DSQRT(A1(J)**2+A2(J)**2))**3)+B1(J)/((A1(J)**2+
     2A2(J)**2)*(A1(J)**2+A2(J)**2+B1(J)**2)))
      FFE17=CCE19*VTHT1**2*U1PX(VVP,VL)*A2(J)*(DATAN(B1(J)/(1.414*A2(J)
     1))/(2.828*A2(J)**3)+B1(J)/((2.*A2(J)**2)*(2.*A2(J)**2+B1(J)**2)))
      FFE18=CCE20*VTHT1**2*U1PX(VVP,VL)*A1(J)*(DATAN(B2(J)/DSQRT(A1(J)
     1**2+A2(J)**2))/((DSQRT(A1(J)**2+A2(J)**2))**3)+B2(J)/((A1(J)**2
     2+A1(J)**2)*(A1(J)**2+A2(J)**2+B1(J)**2)))
      FFE19=CCE21*VTHT1**2*U1PX(VVP,VL)*A2(J)*(DATAN(B2(J)/(1.414*A2(J)
     1))/(2.828*A2(J)**3)+B2(J)/((2.*A2(J)**2)*(2.*A2(J)**2+B2(J)**2)))
      FFE20=CCE22*VTHT1**2*U1(J,X,VL,VVP)*B1P(J)*A1(J)/((A1(J)**2+A2(J)
     1**2+B1(J)**2)**2)
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```
      FFE21=CCE23*VTHT1**2*U1(J,X,VL,VVP)*B1P(J)*A2(J)/((2.*A2(J)**2+
     1B1(J)**2)**2)
      FFE22=CCE24*VTHT1**2*U1(J,X,VL,VVP)*A1(J)/(B1(J)*(A1(J)**2+
     1A2(J)**2+B1(J)**2))
      FFE23=CCE25*VTHT1**2*U1PX(VVP,VL)*A2(J)/(B1(J)*(2.*A2(J)**2+
     1B1(J)**2))
      FFE24=CCE26*VTHT1**2*U1(J,X,VL,VVP)*B1P(J)*A1(J)/(B1(J)**2*(
     1A1(J)**2+A2(J)**2+B1(J)**2))
      FFE25=CCE27*VTHT1**2*U1(J,X,VL,VVP)*B1P(J)*A2(J)/(B1(J)**2*(2.*A
     12(J)**2+B1(J)**2))
      FFE26=CCE28*VTHT1**2*U1(J,X,VL,VVP)*B1P(J)*A1(J)/((A1(J)**2+
     1A2(J)**2+B1(J)**2)**2)
      FFE27=CCE29*VTHT1**2*U1(J,X,VL,VVP)*B1P(J)*A2(J)/((2.*A2(J)**2+
     1B1(J)**2)**2)
      FFE28=CCE30*VTHT1**2*U1(J,X,VL,VVP)*B2P(J)*A1(J)/((A1(J)**2+
     1A2(J)**2+B2(J)**2)**2)
      FFE29=CCE31*VTHT1**2*U1(J,X,VL,VVP)*B2P(J)*A2(J)/((2.*A2(J)**2+
     1B2(J)**2)**2)
      FFE30=CCE32*VTHT1**2*U1PX(VVP,VL)*A1(J)/(B2(J)*(A1(J)**2+
     1A2(J)**2+B2(J)**2))
      FFE31=CCE33*VTHT1**2*U1PX(VVP,VL)*A2(J)/(B2(J)*(2.*A2(J)**2+
     1B2(J)**2))
      FFE32=CCE34*VTHT1**2*U1(J,X,VL,VVP)*B2P(J)*A1(J)/(B2(J)**2*(
     1A1(J)**2+A2(J)**2+B2(J)**2))
      FFE33=CCE35*VTHT1**2*U1(J,X,VL,VVP)*B2P(J)*A2(J)/(B2(J)**2*(
     12.*A2(J)**2+B2(J)**2))
      FFE34=CCE36*VTHT1**2*U1(J,X,VL,VVP)*B2P(J)*A1(J)/((A1(J)**2+
     1A2(J)**2+B2(J)**2)**2)
      FFE35=CCE37*VTHT1**2*U1(J,X,VL,VVP)*B2P(J)*A2(J)/((2.*A2(J)**2+
     1B2(J)**2)**2)
      FFE36=CCE38*VTHT1**2*U1PX(VVP,VL)*A1(J)/(B1(J)*(A1(J)**2+A2(J)**2
     1))
      FFE37=CCE39*VTHT1**2*U1PX(VVP,VL)/(B1(J)*A2(J))
      FFE38=CCE40*VTHT1**2*U1(J,X,VL,VVP)*B1P(J)*A1(J)/(B1(J)**2*(
     1A1(J)**2+A2(J)**2))
      FFE39=CCE41*VTHT1**2*U1(J,X,VL,VVP)*B1P(J)/(B1(J)**2*A2(J))
      FFE40=CCE42*VTHT1**2*U1PX(VVP,VL)*A1(J)/(B2(J)*(A1(J)**2+A2(J)**2
     1))
      FFE41=CCE43*VTHT1**2*U1PX(VVP,VL)/(B2(J)*A2(J))
      FFE42=CCE44*VTHT1**2*U1(J,X,VL,VVP)*B2P(J)*A1(J)/(B2(J)**2*(
     1A1(J)**2+A2(J)**2),
      FFE43=CCE45*VTHT1**2*U1(J,X,VL,VVP)*B2P(J)/(A2(J)*B2(J)**2)
      FFE44=CCE46*VTHT1**2*PYB*A1(J)/(A1(J)**2+A2(J)**2)
      FFE45=CCE47*VTHT1**2*PYB/A2(J)
      FFE46=CCE48*VTHT1**2*A1(J)**3*C*VT1*RHO1/(RHO0**2*PR*DSQRT(
     1A1(J)**2+A2(J)**2)*(A1(J)**2+A2(J)**2))
      FFE47=CCE49*VTHT1**2*RHO1*C*VT1/(PR*RHO0**2)
      FFE48=CCE50*VTHT1*U1(J,X,VL,VVP)**2*B1(J)**2*C*VT1*RHO1/(RHO0**2*
     1(A2(J)**2+2.*B1(J)**2))
      FFE49=CCE51*VTHT1*U1(J,X,VL,VVP)**2*B1(J)*B2(J)*C*VT1*RHO1/(
     1RHO0**2*(A2(J)**2+B1(J)**2+B2(J)**2))
      FFE50=CCE52*VTHT1*U1(J,X,VL,VVP)**2*B2(J)**2*C*VT1*RHO1/(RHO0**2
     1*(A2(J)**2+2.*B2(J)**2))
      FFE51=CCE53*VTHT1**2*DT1*R*A1(J)/(A2(J)**2*(R*VTHT1+ETA*P+R*VTW)*
     1DSQRT(A1(J)**2+A2(J)**2))
      FFE52=CCE54*VTHT1**2*DT1*R/(A2(J)**2*(R*VTHT1+ETA*P+R*VTW))
      FFE53=CCE55*VTHT1*DT1*(ETA*P+R*VTW)/(A2(J)**2*(R*VTHT1+ETA*P+
     1R*VTW))
      FFE54=CCE56*VTHT1**2*U1PX(VVP,VL)*VT1PX*R*A1(J)/(A2(J)**2*DSQRT(
     1A1(J)**2+A2(J)**2)*(R*VTHT1+ETA*P+R*VTW))
      FFE55=CCE57*VTHT1**2*U1(J,X,VL,VVP)*VT1PX*R/(A2(J)**2*(R*VTHT1+
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```
1ETA*P+R*VTW))
 FFE56=CCE58*VTHT1*U1(J,X,VL,VVP)*VT1PX*(ETA*P+R*VTW)/(A2(J)**2*
1(R*VTHT1+ETA*P+R*VTW))
 FFE57=(CCE59*VTHT1**2*U1(J,X,VL,VVP)*DU1(J,X,VL,VVPP,VVP)*R/(R*
1VTHT1+ETA*P+R*VTW))*(A1(J)*DATAN(B1(J)/DSQRT(A1(J)**2+A2(J)**2))/
2(A2(J)**2*DSQRT(A1(J)**2+A2(J)**2))+B1(J)*DATAN(A1(J)/DSQRT(A2(J)
3**2+B1(J)**2))/(A2(J)**2*DSQRT(A2(J)**2+B1(J)**2)))
 FFE58=(CCE60*VTHT1**2*U1(J,X,VL,VVP)*DU1(J,X,VL,VVPP,VVP)*R/(R*
1VTHT1+ETA*P+R*VTW))*(DATAN(B1(J)/(1.414*A2(J)))/(1.414*A2(J)**2)+
2B1(J)*DATAN(A2(J)/DSQRT(A2(J)**2+B1(J)**2))/(A2(J)**2*DSQRT(
3A2(J)**2+B1(J)**2)))
 FFE59=CCE61*VTHT1*U1(J,X,VL,VVP)*DU1(J,X,VL,VVPP,VVP)*(ETA*P+R*
1VTW)*B1(J)/((R*VTHT1+ETA*P+R*VTW)*DSQRT(A2(J)**2+B1(J)**2)*A2(J)
2**2)
 FFE60=(CCE62*VTHT1**2*U1(J,X,VL,VVP)*DU1(4,X,VL,VVPP,VVP)*R/(R*
1VTHT1+ETA*P+R*VTW))*(A1(J)*DATAN(B2(J)/DSQRT(A1(J)**2+A2(J)**2))/
2(A2(J)**2*DSQRT(A1(J)**2+A2(J)**2))+B2(J)*DATAN(A1(J)/DSQRT(
3A2(J)**2+B2(J)**2)))
 FFE61=(CCE63*VTHT1**2*U1(J,X,VL,VVP)*DU1(J,X,VL,VVPP,VVP)*R/(R*
1VTHT1+ETA*P+R*VTW))*(DATAN(B2(J)/(1.414*A2(J)))/(1.414*A2(J)**2)+
2B2(J)*DATAN(A2(J)/DSQRT(A2(J)**2+B2(J)**2))/(A2(J)**2*DSQRT(
3A2(J)**2+B2(J)**2)))
 FFE62=CCE64*VTHT1*U1(J,X,VL,VVP)*DU1(J,X,VL,VVPP,VVP)*(ETA*P+R*VTW
1)*B2(J)/(A2(J)**2*(R*VTHT1+ETA*P+R*VTW)*DSQRT(A2(J)**2+B2(J)**2))
 FFE63=(CCE65*VTHT1**2*U1(J,X,VL,VVP)**2*U1PX(VVP,VL)*R/(R*VTHT1+
1ETA*P+R*VTW))*(A1(J)*DATAN(B1(J)/DSQRT(A1(J)**2+A2(J)**2))/(A2(J)
2**2*DSQRT(A1(J)**2+A2(J)**2))+B1(J)*DATAN(A1(J)/DSQRT(A2(J)**2+
3B1(J)**2))/(A2(J)**2*DSQRT(A2(J)**2+B1(J)**2)))
 FFE64=(CCE66*VTHT1**2*U1(J,X,VL,VVP)**2*U1PX(VVP,VL)*R/(R*VTHT1+
1ETA*P+R*VTW))*(DATAN(B1(J)/(1.414*A2(J)))/(1.414*A2(J)**2)+B1(J)
2*DATAN(A2(J)/DSQRT(A2(J)**2+B1(J)**2))/(A2(J)**2*DSQRT(A2(J)**2+
3B1(J)**2)))
 FFE65=CCE67*VTHT1*U1(J,X,VL,VVP)**2*U1PX(VVP,VL)*(ETA*P+R*VTW)*
1B1(J)/(A2(J)**2*DSQRT(A2(J)**2+B1(J)**2)*(R*VTHT1+ETA*P+R*VTW))
 FFE66=(CCE68*VTHT1**2*U1(J,X,VL,VVP)**2*U1PX(VVP,VL)*R/(R*VTHT1+
1ETA*P+R*VTW))*(A1(J)*DATAN(B2(J)/DSQRT(A1(J)**2+A2(J)**2))/(A2(J)
2**2*DSQRT(A1(J)**2+A2(J)**2))+B2(J)*DATAN(A1(J)/DSQRT(A2(J)**2+
3B2(J)**2))/(A2(J)**2*DSQRT(A2(J)**2+B2(J)**2)))
 FFE67=(CCE69*VTHT1**2*U1(J,X,VL,VVP)**2*U1PX(VVP,VL)*R/(R*VTHT1+
1ETA*P+R*VTW))*(DATAN(B2(J)/(1.414*A2(J)))/(1.414*A2(J)**2)+B2(J)*
2DATAN(A2(J)/DSQRT(A2(J)**2+B2(J)**2))/(A2(J)**2*DSQRT(
3A2(J)**2+B2(J)**2)))
 FFE68=CCE70*VTHT1*U1(J,X,VL,VVP)**2*U1PX(VVP,VL)*(ETA*P+R*VTW)*
1B2(J)/(A2(J)**2*DSQRT(A2(J)**2+B2(J)**2)*(R*VTHT1+ETA*P+R*VTW))
 FFE69=CCE71*VTHT1**2*U1(J,X,VL,VVP)*DU1(J,X,VL,VVPP,VVP)*R*A1(J)/
1(A2(J)**2*DSQRT(A1(J)**2+A2(J)**2)*(R*VTHT1+ETA*P+R*VTW))
 FFE70=CCE72*VTHT1**2*U1(J,X,VL,VVP)*DU1(J,X,VL,VVPP,VVP)*R/(A2(J)
1**2*(R*VTHT1+ETA*P+R*VTW))
 FFE71=CCE73*VTHT1*U1(J,X,VL,VVP)*DU1(J,X,VL,VVPP,VVP)*(ETA*P+
1R*VTW)/(A2(J)**2*(R*VTHT1+ETA*P+R*VTW))
 FFE72=CCE74*VTHT1**2*U1(J,X,VL,VVP)**2*U1PX(VVP,VL)*R*A1(J)/(A2(J)
1**2*DSQRT(A1(J)**2+A2(J)**2)*(R*VTHT1+ETA*P+R*VTW))
 FFE73=CCE75*VTHT1**2*U1(J,X,VL,VVP)**2*U1PX(VVP,VL)*R/(A2(J)**2*
1(R*VTHT1+ETA*P+R*VTW))
 FFE74=CCE76*VTHT1*U1(J,X,VL,VVP)**2*U1PX(VVP,VL)*(ETA*P+R*VTW)/
1(A2(J)**2*(R*VTHT1+ETA*P+R*VTW))
 SUM1=FFE1+FFE2+FFE4+FFE5+FFE6+FFE7+FFE8+FFE9+FFE10+FFE11
 SUM2=FFE12+FFE13+FFE14+FFE15+FFE16+FFE17+FFE18+FFE19+FFE20+FFE21
 SUM3=FFE22+FFE23+FFE24+FFE25+FFE26+FFE27+FFE28+FFE29+FFE30+FFE31
 SUM4=FFE32+FFE33+FFE34+FFE35+FFE36+FFE37+FFE38+FFE39+FFE40+FFE41
 SUM5=FFE42+FFE43+FFE44+FFE45+FFE46+FFE47+FFE48+FFE49+FFE50+FFE51
```

```
      SUM6=FFF52+FF 53+FF 54+FF55+FFE56+FFF57+FFF58+FFE59+FFE60+FF 61
      SUM7=FFE62+FFF63+FFE64+FFE65+FFE66+FFE67+FFE68+FFE69+FFE70+FFE71
      SUM8=FFE72+FFF73+FF 74
      FE2=(SUM1+SUM2+SUM3+SUM4+SUM5+SUM6+SUM7+SUM8)
      RETURN
      END
      FUNCTION U1PX(VVP,VL)
      IMPLICIT REAL*8(A-H),REAL*8(O-Z)
      COMMON OMEGA1,OMEGA2,R,PI,ETA,C,PR,DYB,P1,P2,P3,TW1,TW2
      COMMON TW3,T11,T12,T13,THT11,THT12,THT13,RHOO,RHO11,RHO12,RHO13
      COMMON DRHO11,DRHO12,DRHO13,VTWPX,VTHPX,DTW1,DTW2,DTW3,VL1,VL2
      COMMON VL3,DOMGA,VVP1,VVP2,VVP3,VVPP1,VVPP2,VVPP3,H,DT11,DT12,DT13
      COMMON C4,CC51,C49,CC2,CE5,CE3,CCE3,CCE5,CC5
      COMMON T1PX1,T1PX2,T1PX3,DELTA1,DELTA2,J1,CP
      COMMON DTHT11,DTHT12,DTHT13
      U1PX=0.0
      RETURN
      END
      FUNCTION U1(J,X,VL,VVP)
      IMPLICIT REAL*8(A-H),REAL*8(O-Z)
      COMMON OMEGA1,OMEGA2,R,PI,ETA,C,PR,DYB,P1,P2,P3,TW1,TW2
      COMMON TW3,T11,T12,T13,THT11,THT12,THT13,RHOO,RHO11,RHO12,RHO13
      COMMON DRHO11,DRHO12,DRHO13,VTWPX,VTHPX,DTW1,DTW2,DTW3,VL1,VL2
      COMMON VL3,DOMGA,VVP1,VVP2,VVP3,VVPP1,VVPP2,VVPP3,H,DT11,DT12,DT13
      COMMON C4,CC51,C49,CC2,CE5,CE3,CCE3,CCE5,CC5
      COMMON T1PX1,T1PX2,T1PX3,DELTA1,DELTA2,J1,CP
      COMMON DTHT11,DTHT12,DTHT13
      DIMENSION X(12)
      U1=1.
      RETURN
      END
      FUNCTION DU1(J,X,VL,VVPP,VVP)
      IMPLICIT REAL*8(A-H),REAL*8(O-Z)
      COMMON OMEGA1,OMEGA2,R,PI,ETA,C,PR,DYB,P1,P2,P3,TW1,TW2
      COMMON TW3,T11,T12,T13,THT11,THT12,THT13,RHOO,RHO11,RHO12,RHO13
      COMMON DRHO11,DRHO12,DRHO13,VTWPX,VTHPX,DTW1,DTW2,DTW3,VL1,VL2
      COMMON VL3,DOMGA,VVP1,VVP2,VVP3,VVPP1,VVPP2,VVPP3,H,DT11,DT12,DT13
      COMMON C4,CC51,C49,CC2,CE5,CE3,CCE3,CCE5,CC5
      COMMON T1PX1,T1PX2,T1PX3,DELTA1,DELTA2,J1,CP
      COMMON DTHT11,DTHT12,DTHT13
      DIMENSION X(12)
      DU1=0.0
      RETURN
      END
$ENTRY
         6          6          6          6          17          6          6
4750.      4517.      4058.      3600.      3142.
2358.
0.0        0.002      0.004      0.006      0.008
0.01
1.0        500.       1200.      2000.      2600.
2800.
2.000      2.5000     4.2000     7.4000     10.73000
12.000
4050.      3817.      3358.      2900.      2442.
1658.
2160000.   3312000.   5760000.   2592000.   1440000.
288000.
1056240.   1619568.   2816640.   1267488.   704160
234485.
.1399      .2486      .45 7      .2797      .1762
```

```
.0724
0.0             287500.         -343750.        -425000.        -20166.7
-229916.65
0.0             287500.         -343750.        -425000.        -20166.7
-229916.65
0.0             34.95           -18.10          -69.60          51.90
-192.758
1.0  1.1  1.2  1.3  1.4  1.5  1.6  1.7  1.8  1.9  2.0
$STOP
/*
```